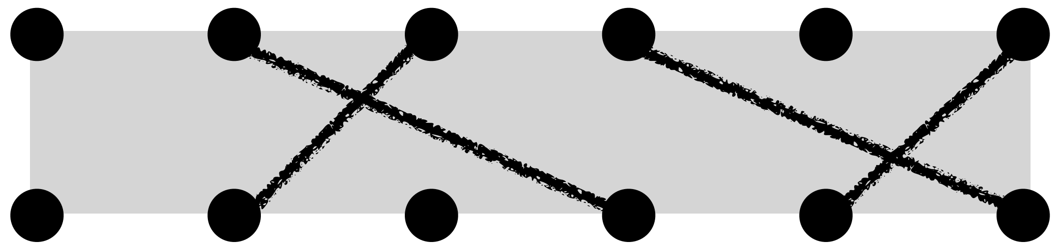
Characters of the rook monoid

Mike Zabrocki (York University)
Joint work with Rosa Orellana (Dartmouth College)
and Alex Wilson (U. of Binghamton)

The rook monoid and representation theory

elements:



$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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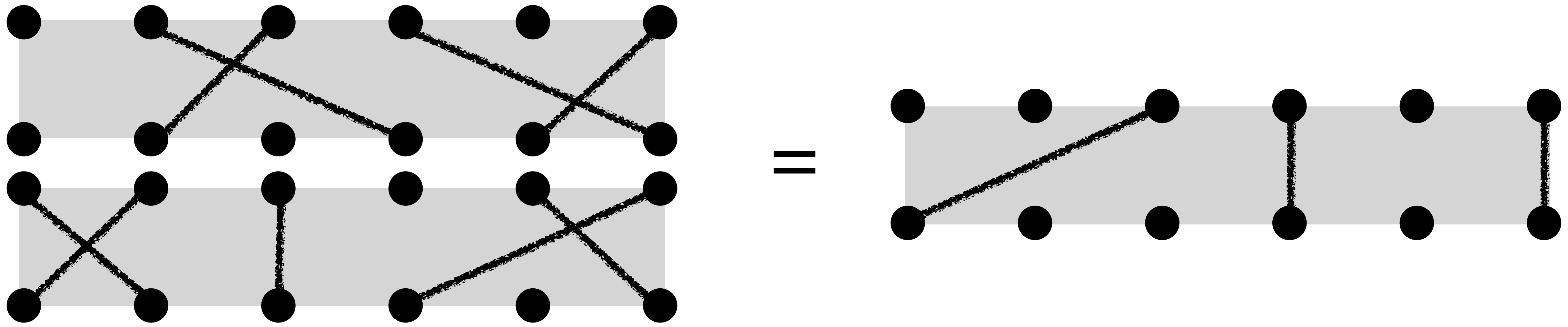

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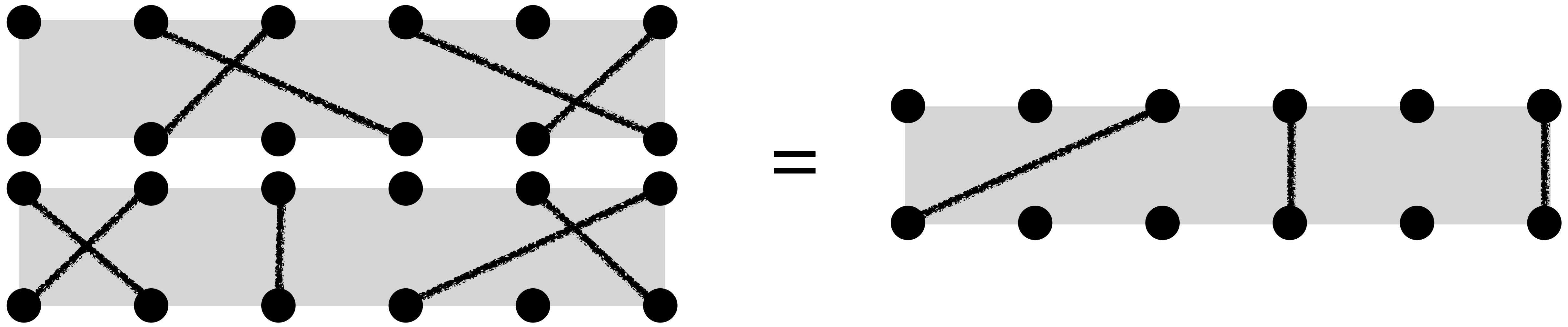

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irreducible modules:

$$V^{(1,1)}_{\mathrm{Rook}_6} = \mathrm{span}$$

$$\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\quad \begin{array}{|c|}\hline 3\\\hline 1\\\hline\end{array}\quad \begin{array}{|c|}\hline 4\\\hline 1\\\hline\end{array}\quad \begin{array}{|c|}\hline 5\\\hline 1\\\hline\end{array}\quad \begin{array}{|c|}\hline 6\\\hline 1\\\hline\end{array}$$

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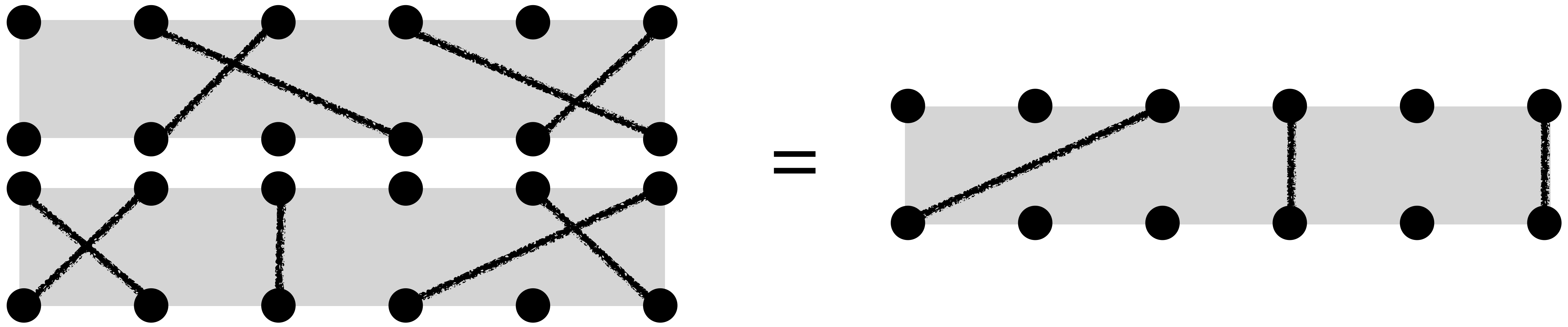

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characters:

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c}
- & 1 & 2 & 1^2 & 3 & 21 & 1^3 & 4 & 31 & 2^2 & 21^2 & 1^4 & 5 & 41 & 32 & 31^2 & 2^21 & 21^3 & 1^5 & 6 & 51 & 42 & 41^2 & 3^2 & 321 & 2^3 & 2^21^2 & 21^4 & 1^6 \\ \hline
0 & 0 & -1 & 1 & 0 & -1 & 3 & 0 & 0 & -2 & 0 & 6 & 0 & 0 & -1 & 1 & -2 & 2 & 10 & 0 & 0 & -1 & 1 & 0 & -1 & -3 & -1 & 5 & 15
\end{array}\right)$$

$$\tilde{x}_{11} = e_2 = \frac{p_{11}}{2} - \frac{p_2}{2} = s_{11}$$

Classical application of symmetric functions to representation theory

$Gl_n \times S_k$ acts on the tensor space

$$Gl_n \;\text{———}\; V_n^{\otimes k} \;\text{———}\; S_k$$

$$v_{i_1} \otimes v_{i_2} \otimes v_{i_3} \otimes v_{i_4} \otimes v_{i_5} \otimes v_{i_6} \otimes v_{i_7} \otimes v_{i_8} \otimes v_{i_9}$$

$$A \in \overline{Gl_n} \downarrow$$

$$Av_{i_1} \otimes Av_{i_2} \otimes Av_{i_3} \otimes Av_{i_4} \otimes Av_{i_5} \otimes Av_{i_6} \otimes Av_{i_7} \otimes Av_{i_8} \otimes Av_{i_9}$$

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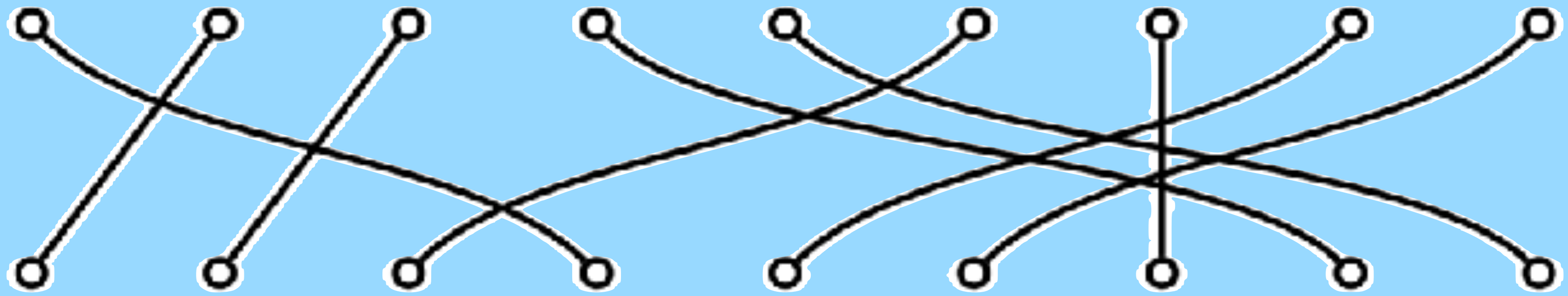

$\in S_9$

$$v_{i_2} \otimes v_{i_3} \otimes v_{i_6} \otimes v_{i_1} \otimes v_{i_8} \otimes v_{i_9} \otimes v_{i_7} \otimes v_{i_4} \otimes v_{i_5}$$

$$V_n^{\otimes k} = \bigoplus_{\lambda \vdash k} V_{Gl_n}^{\lambda} \otimes W_{S_k}^{\lambda}$$

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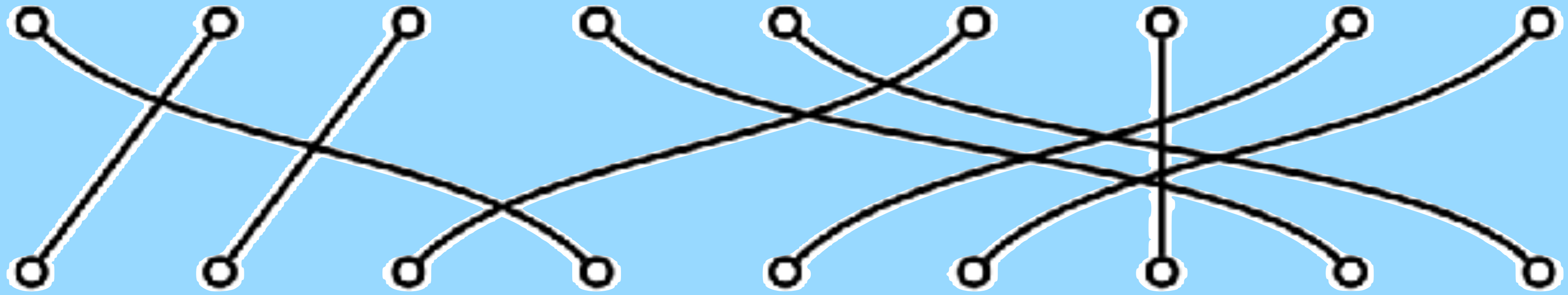

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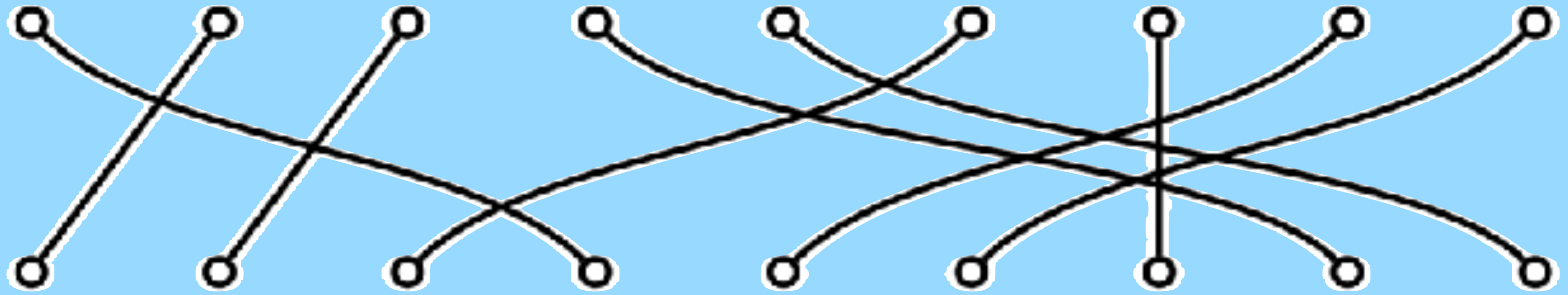

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$$(x_1 + x_2 + \cdots + x_n)^k = \sum_{\lambda} f^{\lambda} s_{\lambda}(x_1, x_2, \dots, x_k)$$

Classical application of symmetric functions to representation theory

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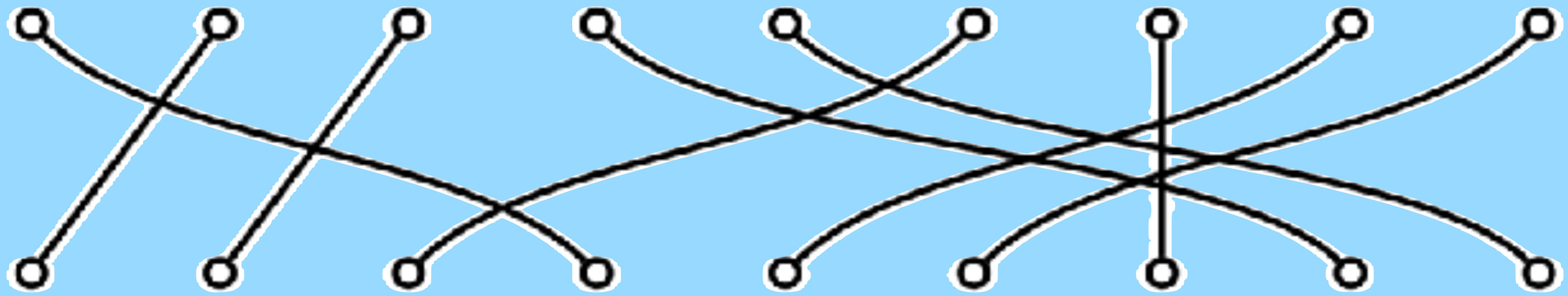

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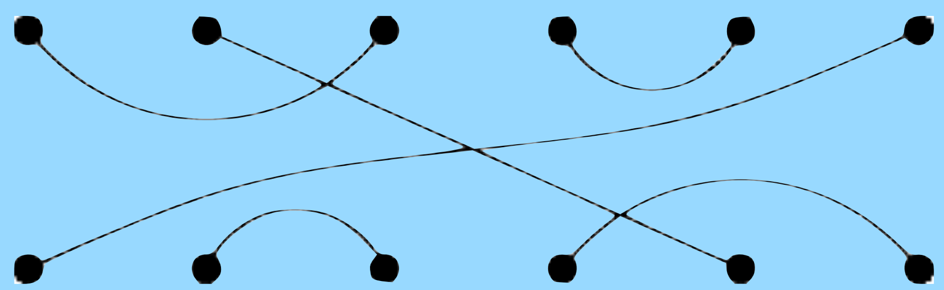
Representation theory	$V_n^{\otimes k} = \bigoplus_{\lambda \vdash k} V^\lambda_{Gl_n} \otimes W^\lambda_{S_k}$
Combinatorics	$n^k = \#\{(P,Q) \in SSYT^k_{1\ldots n} \times SYT^k : shape(P) = shape(Q)\}$
Symmetric functions	$(x_1 + x_2 + \cdots + x_n)^k = \sum_\lambda f^\lambda s_\lambda(x_1, x_2, \ldots, x_k)$
	$p_\mu(x_1, x_2, \ldots, x_n) = \sum_\lambda \chi^\lambda_{S_k}(\mu) s_\lambda(x_1, x_2, \ldots, x_n)$

Still classical, but less well known application of symmetric functions

$$O_n \quad \text{———} \quad V_n^{\otimes k} \quad \text{———} \quad \mathrm{Brauer}_k(n)$$



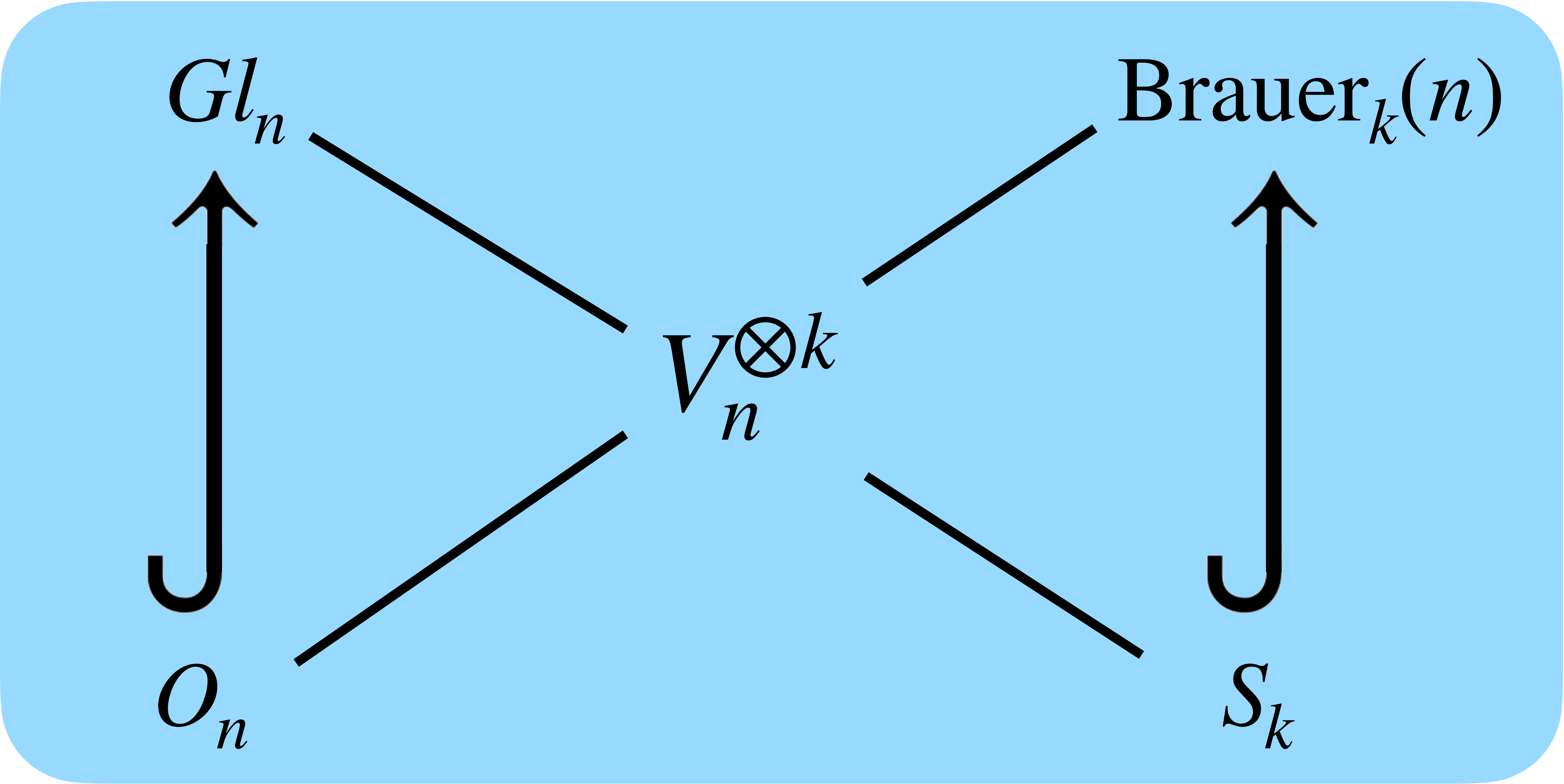
Schur-Weyl duality R. Brauer (1937)

Weyl character formula - characters of the irreducible O_n modules

Koike-Terada 1990's defined basis $\quad o_\lambda(x_1, x_2, \ldots, x_n)$

Sundaram 1990's orthogonal tableaux models

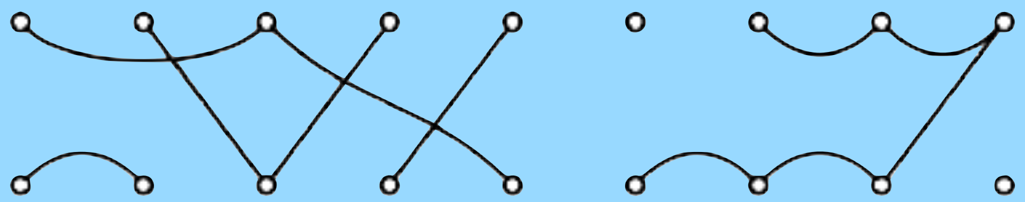
See-saw pair - restriction is the same on both sides



$$V^{\lambda}_{Gl_n} = \bigoplus_{\mu} (V^{\mu}_{O_n})^{d_{\lambda\mu}} \qquad W^{\mu}_{\mathrm{Brauer}_k} = \bigoplus_{\lambda} (W^{\lambda}_{S_k})^{d_{\lambda\mu}} \qquad s_{\lambda}(x_1, \ldots, x_n) = \sum_{\mu} d_{\lambda\mu} o_{\mu}(x_1, \ldots, x_n)$$

partition algebra

$$S_n \;\text{———}\; V_n^{\otimes k} \;\text{———}\; \mathrm{P}_k(n)$$



Schur-Weyl duality Martin and Jones in 1990’s

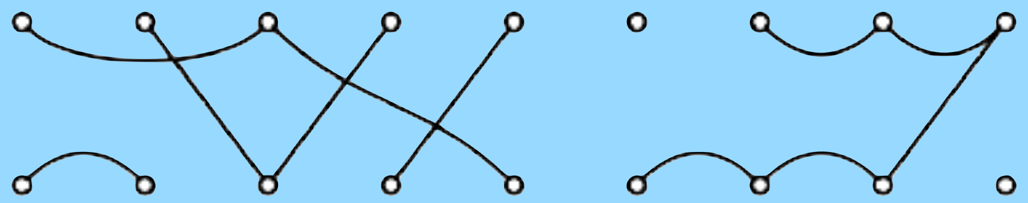
Orellana and Zabrocki (2016) - character basis as symmetric functions $\tilde{s}_\lambda(x_1, x_2, \ldots, x_n)$

Assaf-Speyer - transition $\tilde{s}_\lambda \to s_\mu$ is alternating with respect to degree

vascillating tableaux - Halverson, Ram, Lewandowski 1990’s
set valued tableaux - Benkart, Halverson, Harman + COSSZ around 2020
Colmenarejo, Orellana, Saliola, Schilling, Z.

partition algebra

S_n ——— $V_n^{\otimes k}$ ——— $\mathrm{P}_k(n)$



Schur-Weyl duality Martin and Jones in 1990's

much more to understand here!

Orellana and Zabrocki (2016) - character basis as symmetric functions $\tilde{s}_\lambda(x_1, x_2, \ldots, x_n)$

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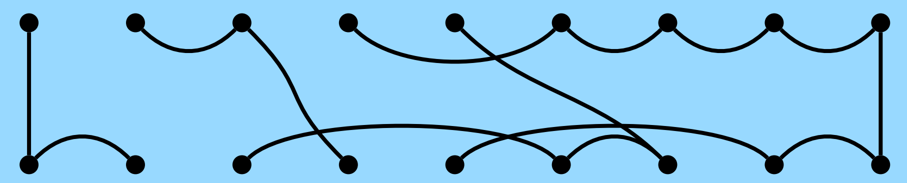
propagating partition algebra

$$\mathrm{Rook}_n \;\text{——}\; V_n^{\otimes k} \;\text{———}\; \mathrm{Prop}_k$$



Schur-Weyl duality

Kudryavtseva, Ganna; Mazorchuk, Volodymyr (2008).
"Schur–Weyl dualities for symmetric inverse semigroups”.

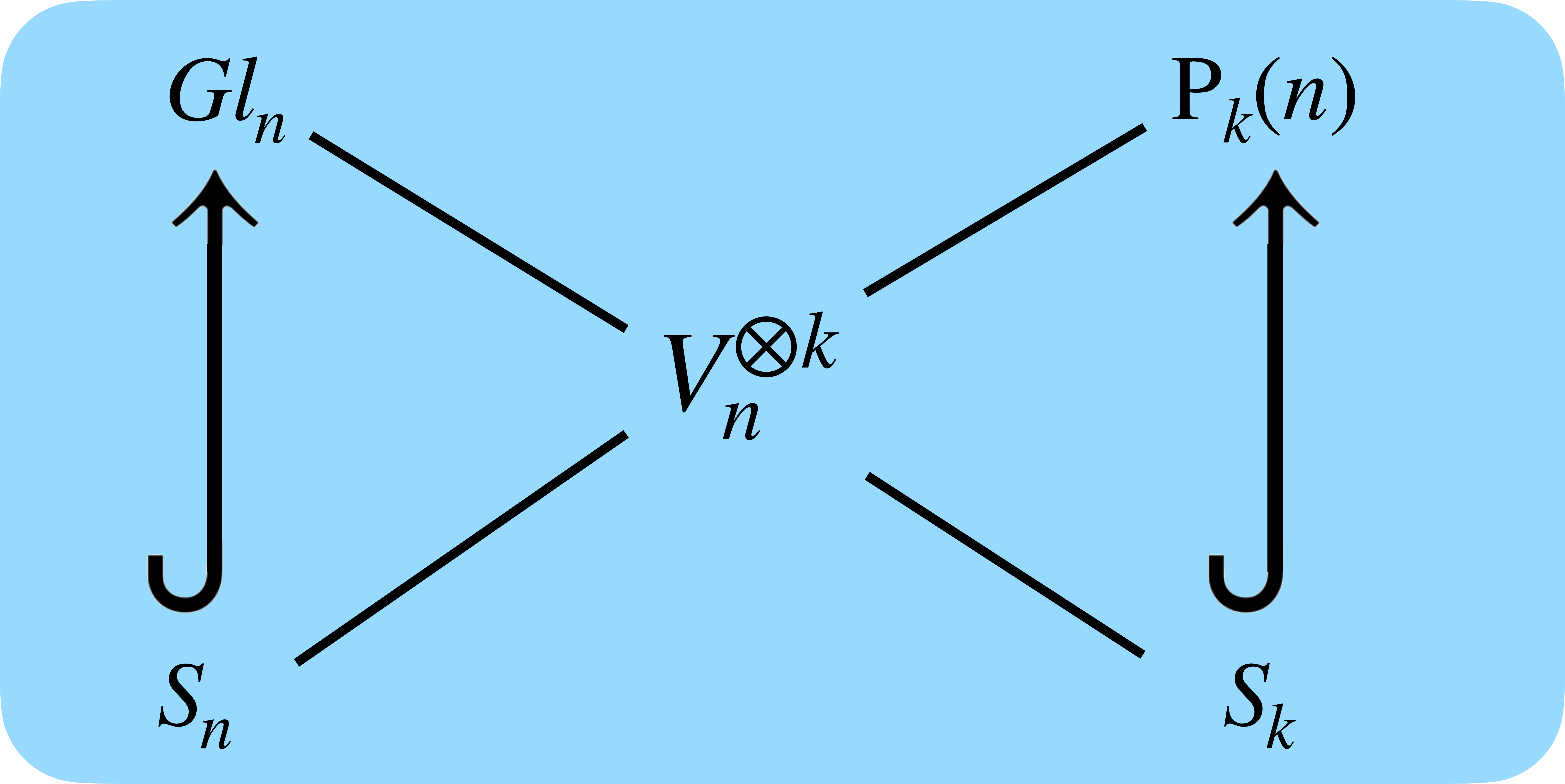
rook Kronecker

Mishra, Ashish; Srivastava, Shraddha (2021).
"Jucys–Murphy elements of partition algebras for the rook monoid"

Assaf-Speyer (2018) (+ Orellana-Zabrocki 2019 for comment of relation to rook monoid)

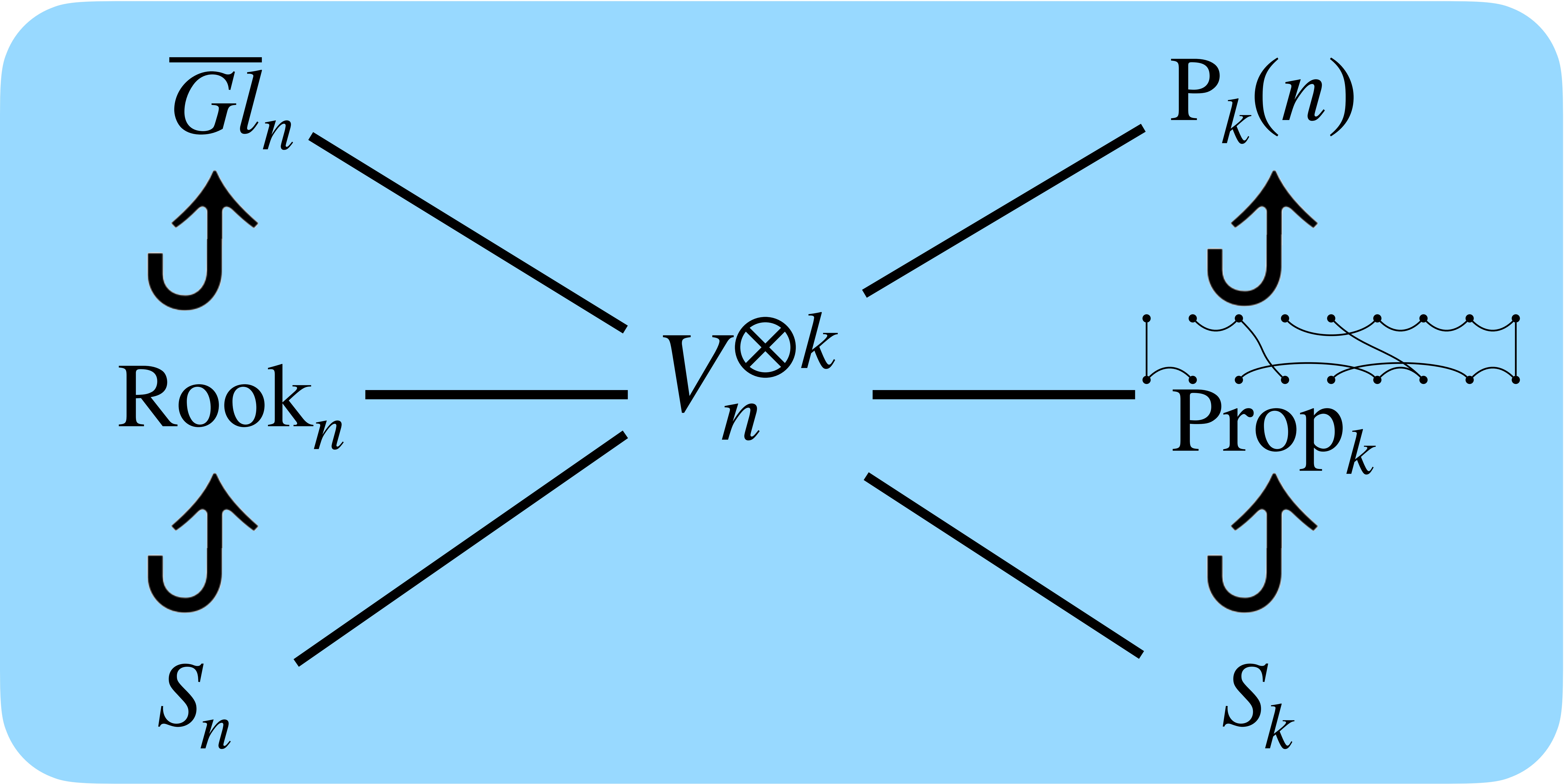
$$\tilde{x}_\lambda(x_1, \ldots, x_n) = \sum_{\lambda/\mu \text{ is h-strip}} \tilde{s}_\mu(x_1, \ldots, x_n)$$

See-saw pair - restriction is the same on both sides



Focus of this work: determine how $\tilde{x}_\lambda(x_1, x_2, \ldots, x_n)$ relates to this picture.

See-saw pair - restriction is the same on both sides



Answer: $\tilde{x}_\lambda(x_1, x_2, \ldots, x_n)$ are the irreducible characters of the rook monoid (partial permutation matrices)

6 characterizations of the $\tilde{x}_\lambda(x_1, \ldots, x_n)$

one: power sum expansion

$$\tilde{x}_\lambda := \sum_{\gamma \vdash n} \chi^\lambda_{S_n}(\gamma) \frac{\overline{\mathbf{p}}_\gamma}{z_\gamma} \qquad \text{with} \qquad \overline{\mathbf{p}}_\lambda := \prod_{i \geq 1} i^{m_i} \prod_{n=0}^{m_i - 1} \left(\left(\frac{1}{i} \sum_{d|i} \boldsymbol{\mu}(i/d) p_d \right) - n \right)$$

where $\lambda = (1^{m_1} 2^{m_2} \cdots \ell^{m_\ell})$ and $\boldsymbol{\mu}$ is the Möbius function on integers.

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two: dual basis

The set $\{\tilde{x}_\lambda\}$ is the graded dual basis to $\{s_\lambda[\Omega - 1]\}$ with respect to the scalar product on symmetric functions.

$$\Omega = 1 + \sum_{n \geq 1} h_n$$

6 characterizations of the $\tilde{x}_\lambda(x_1, \ldots, x_n)$

three: evaluations are symmetric group characters

$$|\mu| = |\lambda|, \; then \; \tilde{x}_\lambda[\Xi_\mu] = \chi^\lambda_{S_n}(\mu) \qquad\qquad |\mu| < |\lambda|, \; \tilde{x}_\lambda[\Xi_\mu] = 0$$

$f[\Xi_\mu]$ evaluation f at eigenvalues of rook monoid matrix with cycle structure μ

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$f[\Xi_\mu]$ evaluation f at eigenvalues of rook monoid matrix with cycle structure μ

four: evaluations are rook characters

$$|\mu| \geq |\lambda|,\ \tilde{x}_\lambda[\Xi_\mu] = \chi^\lambda_{R_n}(\mu)$$

characters of rook monoid - Solomon (2002)

6 characterizations of the $\tilde{x}_\lambda(x_1, \ldots, x_n)$

five: structure coefficients are the smash product or "Heisenberg" product

The set $\{\tilde{x}_\lambda\}$ *is the unique basis whose structure coefficients are the same as the smash (or Heisenberg) product coefficients on Schur functions and* $\tilde{x}_{1^r} = e_r$ *for all* $r \geq 1$.

smash product interpolates LR & Kronecker due to Aguiar-Ferrer-Moreira (2004)

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six: transition coefficients from homogeneous basis count multiset tableaux

$$h_\mu = \sum_\lambda \mathcal{K}_{\lambda\mu} \tilde{x}_\lambda$$

$$\begin{array}{|c|c} \hline 13 & \\ \hline 112 & \multicolumn{1}{c|}{23} \\ \hline \end{array}$$

example with $\mu = (3, 2, 2)$ $\lambda = (2, 1)$

$\mathcal{K}_{\lambda\mu}$ *is the number of multiset tableaux of shape* λ *and content* μ

TODO:

- monomial and Schur expansions of $\tilde{x}_\lambda$

- structure coefficients of $\tilde{x}_\lambda \cdot \tilde{x}_\mu = \sum_{\gamma \vdash |\lambda| + |\mu|} c^{\gamma}_{\lambda\mu} \tilde{x}_\gamma + \cdots + \sum_{\gamma} g_{\gamma\lambda\mu} \tilde{x}_\gamma$

- combinatorics of multiset tableaux
(e.g. crystals, notion of lattice, etc.)