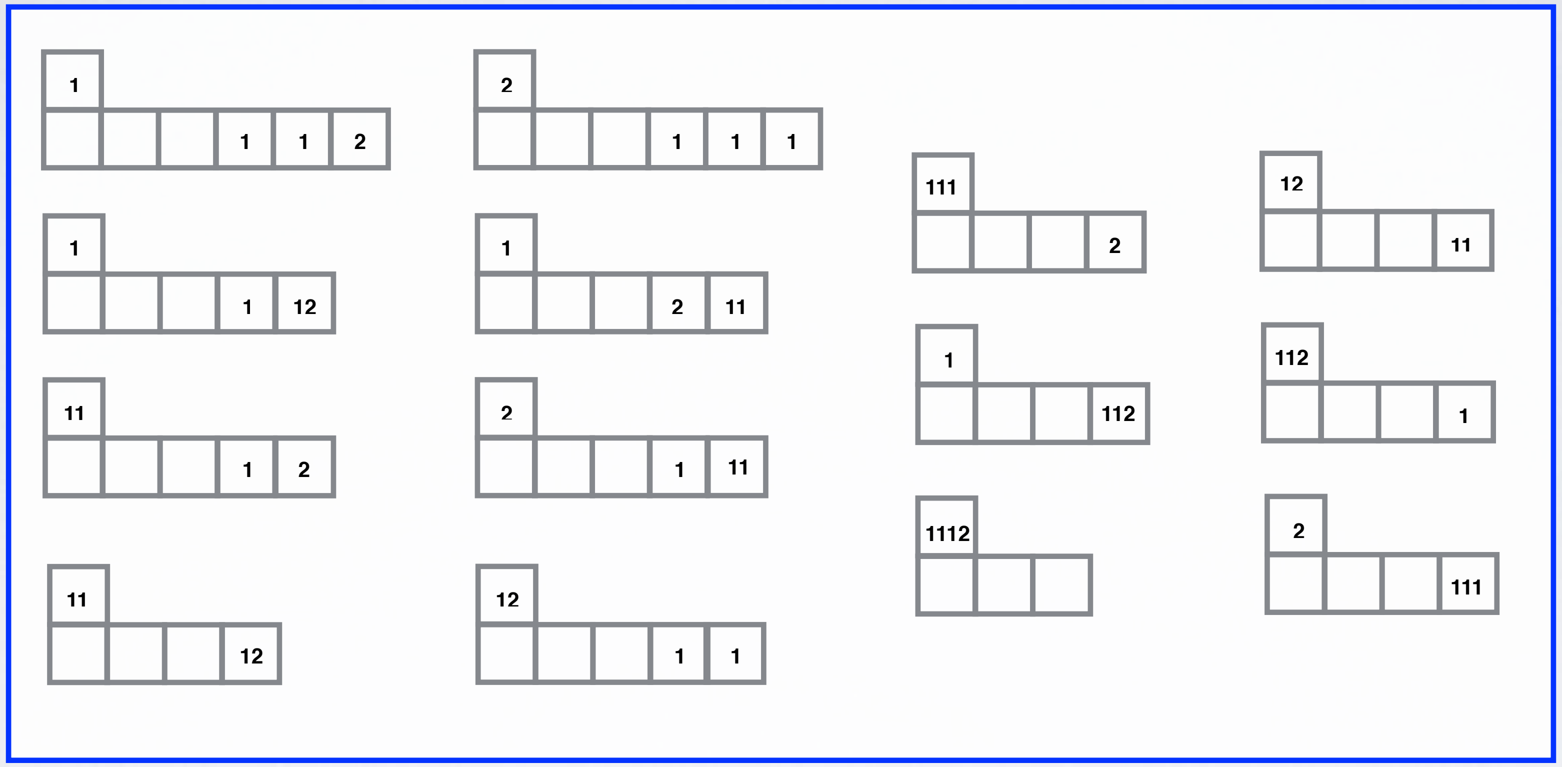

On symmetric group and partition algebra characters

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Schur-Weyl Duality

V is an n-dimensional Gl_n module

$$A \in Gl_n \qquad\qquad (v_i)A = \sum_{j=1}^{n} a_{ij} v_j$$

$$T^k(V) = \underbrace{V \otimes V \otimes \cdots \otimes V}_{\mathbb{S}_k \text{ action}} \simeq \sum_{\lambda \vdash k} S^\lambda \otimes M^\lambda$$

centralizer algebra $\mathbf{k}\mathbb{S}_k$

Frobenius image of character

$$s_\lambda = \frac{1}{k!} \sum_{\sigma \in \mathbb{S}_k} char_{S^\lambda}(\sigma) p_{type(\sigma)}$$

diagonal Gl_n action

irreducible characters

Schur functions

$$s_\lambda(z_1, z_2, \ldots, z_n) = char_{M^\lambda}(A)$$

Schur-Weyl Duality

V is an n-dimensional $\begin{matrix} O_{2n} \\ Sp_{2n} \\ O_{2n+1} \end{matrix}$ module

$$A \in \begin{matrix} O_{2n} \\ Sp_{2n} \\ O_{2n+1} \end{matrix} \qquad (v_i)A = \sum_{j=1}^{n} a_{ij} v_j$$

$$T^k(V) = \underbrace{V \otimes V \otimes \cdots \otimes V}_{\text{Brauer algebra action}} \simeq \sum_{\lambda \vdash k} S^\lambda \otimes M^\lambda$$

centralizer is Brauer algebra

diagonal $\begin{matrix} O_{2n} \\ Sp_{2n} \\ O_{2n+1} \end{matrix}$ action

irreducible characters

universal character functions

$$\left.\begin{matrix} (\Omega[-s_2]^\perp s_\lambda)(z_1, z_2, \ldots, z_n, z_1^{-1}, z_2^{-1}, \ldots, z_n^{-1}) \\ (\Omega[-s_{11}]^\perp s_\lambda)(z_1, z_2, \ldots, z_n, z_1^{-1}, z_2^{-1}, \ldots, z_n^{-1}) \\ (\Omega[-s_2]^\perp s_\lambda)(z_1, z_2, \ldots, z_n, z_1^{-1}, z_2^{-1}, \ldots, z_n^{-1}, 1) \end{matrix}\right\} = char_{M^\lambda}(A)$$

Schur-Weyl Duality

V is an n-dimensional $\mathbb{S}_n$ module

$A \in \mathbb{S}_n$ $\qquad$ $(v_i)A = \sum_{j=1}^{n} a_{ij} v_j$

$$T^k(V) = \underbrace{V \otimes V \otimes \cdots \otimes V}_{\text{partition algebra action}} \simeq \sum_{\lambda \vdash k} S^\lambda \otimes M^\lambda$$

centralizer is partition algebra

diagonal $\mathbb{S}_n$ action

irreducible characters

??????

The irreducible characters of the symmetric group form a basis of the symmetric functions.

$$\Xi_r := 1, \zeta_r, \zeta_r^2, \ldots, \zeta_r^{r-1} \qquad \zeta_r = e^{2\pi i/r}$$

$$\Xi_\mu := \Xi_{\mu_1}, \Xi_{\mu_2}, \ldots, \Xi_{\mu_{\ell(\mu)}}$$

eigenvalues of a permutation matrix with cycle structure μ

$$\tilde{s}_\lambda(\Xi_\mu) = \chi^{(|\mu|-|\lambda|,\lambda)}(\mu)$$

The irreducible characters of the symmetric group are built from induced trivial characters

multi-set partitions of a multi-set

$$h_\nu = \sum_{\pi \Vdash \{1^{\nu_1}, \ldots, \ell^{\nu_\ell}\}} \tilde{h}_{\tilde{m}(\pi)}$$

$\tilde{m}(\pi) =$ vector of number of times sets repeat in multi-set partition

$$\tilde{h}_\lambda(\Xi_\mu) = \langle h_{(|\mu|-|\lambda|,\lambda)}, p_\mu \rangle$$

Theorem

The coefficient of $\tilde{h}_\lambda$ in h_μ

is the number of multiset partitions of a multiset

$\{\{1,1\},\{1,2\}\}$

$\{\{1,1,1\},\{2\}\}$

$\{\{1\},\{1,1,2\}\}$

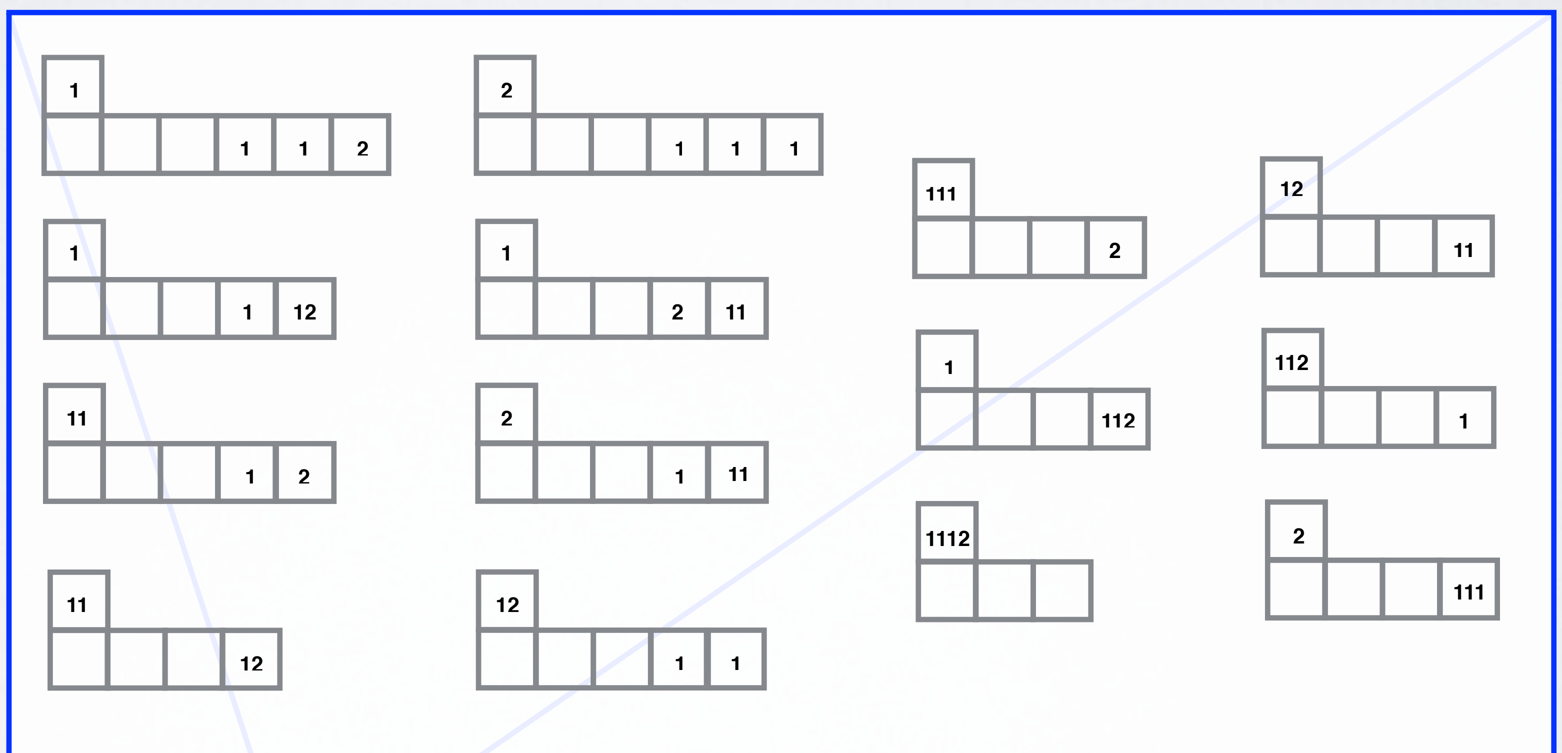
$\{\{1\},\{1\},\{1,2\}\}$

$$h_{31} = \tilde{h}_1 + 3\tilde{h}_{11} + \tilde{h}_{111} + \tilde{h}_{21} + \tilde{h}_{31}$$

Theorem

The coefficient of $\tilde{s}_\lambda$ in h_μ

is the number of column strict tableaux of shape (r, λ) and content μ whose entries are multisets



$$h_{31} = 7\tilde{s}_{()} + 14\tilde{s}_1 + 8\tilde{s}_{11} + \tilde{s}_{111} + 10\tilde{s}_2$$
$$+4\tilde{s}_{21} + 4\tilde{s}_3 + \tilde{s}_{31} + \tilde{s}_4$$

Structure coefficients are Kronecker

For $\lambda, \mu \vdash n$ where n is sufficiently large

$$\tilde{s}_{\overline{\lambda}}\tilde{s}_{\overline{\mu}} = \sum_{\nu \vdash n} k_{\lambda\mu\nu}\tilde{s}_{\overline{\nu}}$$

where the $k_{\lambda\mu\nu}$ are the coefficients in the Kronecker product

$$s_\lambda * s_\mu = \sum_{\nu \vdash n} k_{\lambda\mu\nu} s_\nu$$

Useful tool in expressions for characters of representations that depend on n of S_n and stabilize as the size of n increases

Kronecker

plethysm

inner plethysm

certain quotients

Kronecker product of complete/homogeneous

$$h_\lambda * h_\mu = \sum_{\substack{M: row(M)=\lambda \\ col(M)=\mu}} h_{read(M)}$$

Example:

$$\lambda = (3,2) \qquad \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$$

$$\mu = (4,1)$$

$$h_{32} * h_{41} = h_{311} + h_{221}$$

Usual product of induced trivial character

$$\tilde{h}_{\tilde{m}(\pi)}\tilde{h}_{\tilde{m}(\tau)} = \sum_{\gamma\in\pi\#\tau} \tilde{h}_{\tilde{m}(\gamma)}$$

$$\pi\#\tau = \text{set of } \{S_{i_1},\ldots,S_{i_k},T_{j_1},\ldots,T_{j_r},S_{i'_1}\cup T_{j'_1},\ldots,S_{i'_r}\cup T_{j'_r}\}$$

$$\pi = \{\{1\},\{1\}\}$$

$$\tau = \{\{2\}\}$$

$$\tilde{h}_2\tilde{h}_1 = \tilde{h}_{11} + \tilde{h}_{21}$$

$\pi\#\tau$

$\{\{1\},\{1,2\}\}$

$\{\{1\},\{1\},\{2\}\}$

Implementation in Sage 6.10

```
sage: Sym = SymmetricFunctions(QQ)
sage: st = Sym.irreducible_symmetric_group_character()
sage: st
Symmetric Functions over Rational Field in the irreducible symmetric group character basis
sage: s = Sym.Schur()
```

```
sage: s(st[3,2]) # expand irreducible character in the Schur basis
3*s[1] - 6*s[1, 1] - 6*s[2] + 3*s[1, 1, 1] + 8*s[2, 1] + 4*s[3] - s[2, 1, 1] - 2*s[2, 2] - 3*s[3, 1]
- s[4] + s[3, 2]
```

```
sage: st(s[3,2]) # expand Schur function in the irreducible basis
4*st[] + 10*st[1] + 8*st[1, 1] + 11*st[2] + 2*st[1, 1, 1] + 8*st[2, 1] + 6*st[3] + st[2, 1, 1]
+ 2*st[2, 2] + 3*st[3, 1] + st[4] + st[3, 2]
```

```
sage: st[2]*st[2,1]
st[1] + 2*st[1, 1] + 2*st[2] + 2*st[1, 1, 1] + 4*st[2, 1] + 2*st[3] + st[1, 1, 1, 1] + 3*st[2, 1, 1]
+ 2*st[2, 2] + 3*st[3, 1] + st[4] + st[2, 2, 1] + st[3, 1, 1] + st[3, 2] + st[4, 1]
```

```
sage: s[7,2].kronecker_product(s[6,2,1])
s[8, 1] + 2*s[7, 2] + 2*s[7, 1, 1] + 2*s[6, 1, 1, 1] + 4*s[6, 2, 1] + 2*s[6, 3] + s[5, 1, 1, 1, 1]
+ 3*s[5, 2, 1, 1] + 2*s[5, 2, 2] + 3*s[5, 3, 1] + s[5, 4] + s[4, 2, 2, 1] + s[4, 3, 1, 1]
+ s[4, 3, 2] + s[4, 4, 1]
```

Partition algebra characters

transition coefficients from p_μ to $\tilde{s}_\lambda$ are partition algebra characters

$$p_\mu = \sum_{|\lambda| \leq |\mu|} \chi_{P_{|\mu|}}^{(|\mu|-|\lambda|,\lambda)}(d_\mu)\tilde{s}_\lambda$$

```
sage: st(p[2,1])
3*st[] + 4*st[1] - st[1, 1, 1] + 2*st[2] + st[3]
```

(Halverson 2000)

TABLE II
Characters of $P_3(x)$

	∅	(1)	(2)	(1^2)	(3)	(2, 1)	(1^3)
∅	x^3	x^2	$2x$	$2x$	2	3	5
(1)	0	x^2	x	$3x$	1	4	10
(2)	0	0	x	x	0	2	6
(1^2)	0	0	$-x$	1	0	0	6
(3)	0	0	0	0	1	1	1
(2, 1)	0	0	0	0	−1	0	2
(1^3)	0	0	0	0	1	−1	1

Character polynomials give power sum expansion

$$p_k[\Xi_{1^{m_1}2^{m_2}\dots r^{m_r}}] = \sum_{d|k} d m_d$$

$$\tilde{s}_\lambda\Big|_{p_k \to \sum_{d|k} d m_d} = q_\lambda(m_1, m_2, m_3, \dots)$$

$$k m_k = \sum_{d|k} \mu(k/d) p_d$$

$$q_\lambda\left(p_1, \frac{p_2 - p_1}{2}, \frac{p_3 - p_1}{3}, \dots\right) = \tilde{s}_\lambda$$

Character polynomials give power sum expansion

$$\tilde{s}_\lambda = \sum_{\gamma \vdash |\lambda|} \chi^\lambda(\gamma) \frac{\mathbf{p}_\gamma}{z_\gamma}$$

where

$$\mathbf{p}_{i^r} = \sum_{k=0}^{r} (-1)^{r-k} i^k \binom{r}{k} \left(\frac{1}{i} \sum_{d|i} \mu(i/d) p_d \right)_k$$

$$\mathbf{p}_\gamma := \prod_{i \geq 1} \mathbf{p}_{i^{m_i(\gamma)}}$$