Open Problems in Combinatorial Representation Theory

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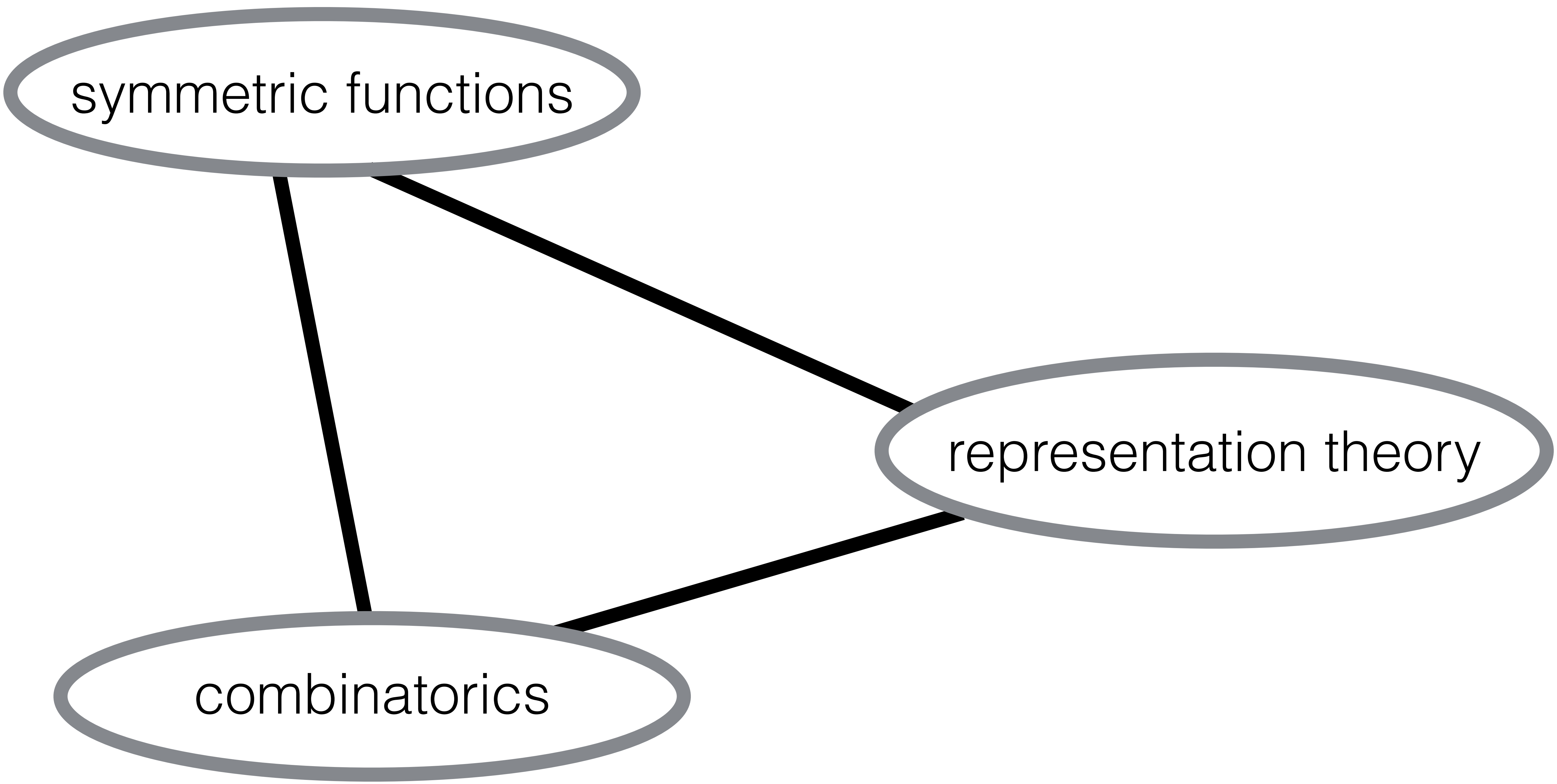
joint work with Rosa Orellana



Combinatorial representation theory

Basic outline of this talk

Representation theory 101, characters and
how they relate to symmetric functions and algebraic combinatorics

List 3-4 of the ‘first question’ open problems in combinatorial representation theory
Kronecker, restriction, inner/outer plethysm

Show how to compute examples in Sage

Lessons learned

A representation refers to a **homomorphism** from a group or algebra into the ring of matrices. Equivalently, we can think of a representation as an **action** of the group or algebra on a vector space.

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$$\phi_i(id) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\phi_1(a) = \begin{bmatrix} 3 & 0 & 4 \\ 0 & 1 & 0 \\ -2 & 0 & -3 \end{bmatrix} \qquad \phi_2(a) = \begin{bmatrix} -5 & 4 & -4 \\ -4 & 3 & -4 \\ 2 & -2 & 1 \end{bmatrix} \qquad \phi_3(a) = \begin{bmatrix} -3 & 0 & -4 \\ -2 & -1 & -4 \\ 2 & 0 & 3 \end{bmatrix}$$

Two representations $\psi(g), \rho(g)$ are equivalent if there is a matrix A such that $A\psi(g)A^{-1} = \rho(g)$

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Fun fact: the character *characterizes* the representation

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Ring of symmetric functions

polynomials in variables $x_1, x_2, \ldots, x_n$ such that

$$f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = f(x_1, x_2, \ldots, x_n)$$

OR

polynomials in generators $p_1, p_2, p_3, \ldots$

p_k represents $x_1^k + x_2^k + \cdots + x_n^k$

```
sage: Sym = SymmetricFunctions(QQ); Sym
Symmetric Functions over Rational Field
sage: p = Sym.powersum(); p
Symmetric Functions over Rational Field in the powersum basis
```

Examples of symmetric functions in Sage

```
sage: Sym = SymmetricFunctions(QQ); Sym
Symmetric Functions over Rational Field
sage: p = Sym.p(); p
Symmetric Functions over Rational Field in the powersum basis
sage: s = Sym.s(); s
Symmetric Functions over Rational Field in the Schur basis

sage: (p[1,1]/2+p[2]/2)*p[3]
1/2*p[3, 1, 1] + 1/2*p[3, 2]

sage: p(s[2,2])
1/12*p[1, 1, 1, 1] + 1/4*p[2, 2] - 1/3*p[3, 1]

sage: s[2]*s[2]
s[2, 2] + s[3, 1] + s[4]

sage: s(p[2,2])
-s[1, 1, 1, 1] - s[2, 1, 1] + s[3, 1] + s[4]
```

Characters **are** symmetric functions

representations	**symmetric functions**
compose/decompose direct sum of matrices	sum
product/inverse of matrices	product
tensor product of matrices	Kronecker product
restrict/induct	composition (plethysm)
composition	coproduct

Representations[1] can be broken down into irreducible components

irreducible representations of certain groups
the characters are known and forms a basis for
the space of symmetric functions

general linear	Gl_n	Schur functions	s_λ
orthogonal	O_n	"universal characters"	o_λ
symplectic	Sp_n		sp_λ

1. certain restrictions apply

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Fun fact: a symmetric function is a positive linear combination of irreducibles iff it is a character of a representation

1. certain restrictions apply

$$tr(A \otimes B) = tr(A)tr(B)$$

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix}$$

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A: $c^{\nu}_{\lambda\mu}$ Littlewood-Richardson rule
- combinatorial description

$$s_\lambda s_\mu = \sum_\nu c^{\nu}_{\lambda\mu} s_\nu$$

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Note: rule for "universal characters" similar

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Q: if you take tensor of two irreducible Sn reps, how do they decompose?

Open problem #1:

A:

this is called the Kronecker product problem
no satisfying rule seems to exist for this decomposition except in special cases

Representations can be broken down into irreducible components

irreducible representations of certain groups
the characters are known and forms a basis for
the space of symmetric functions

general linear	Gl_n	Schur functions	s_λ
orthogonal	O_n	"universal characters"	o_λ
symplectic	Sp_n		sp_λ
symmetric	S_n	"irreducible symmetric group character"	$\tilde{s}_\lambda$

the irreducible character basis encodes combinatorics of multi-set valued tableaux

3			
23			
1	12		
			11

the structure coefficients of this basis are the (reduced) Kronecker coefficients

$$S_n \subseteq Gl_n$$

The symmetric group realized as permutation matrices sits inside of the general linear group

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Open problem #2:

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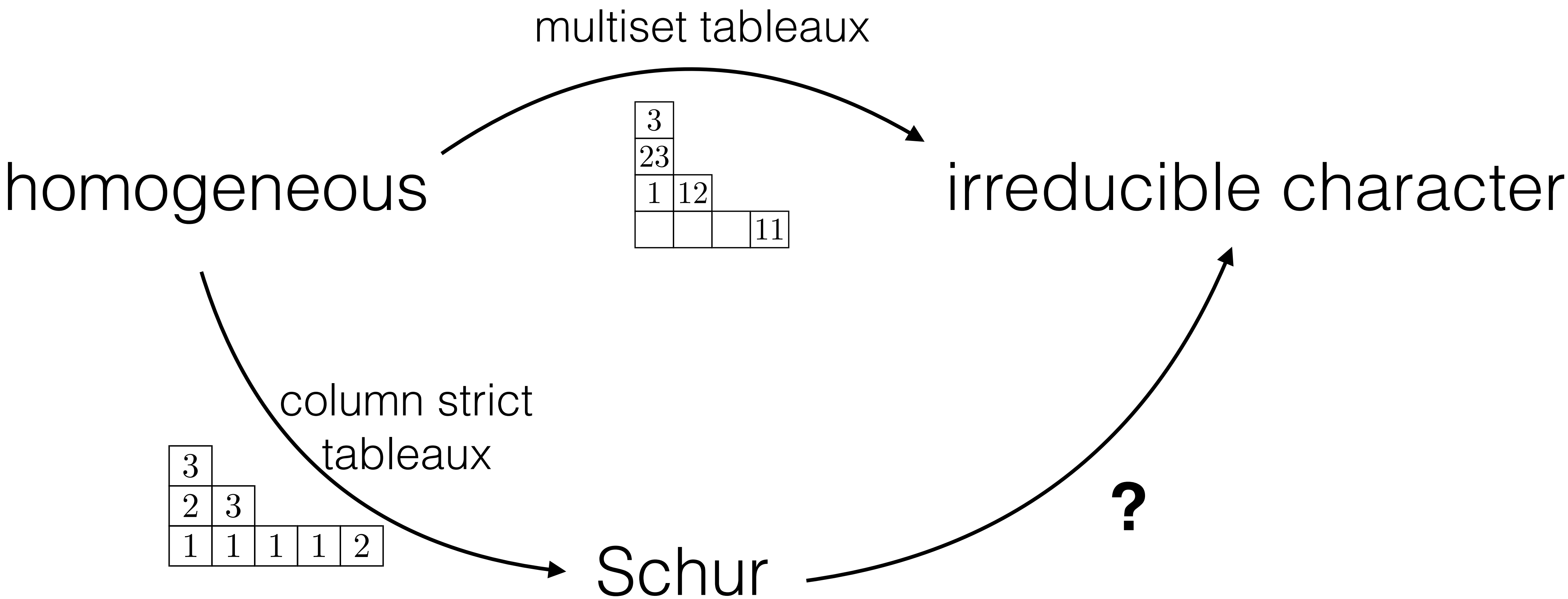
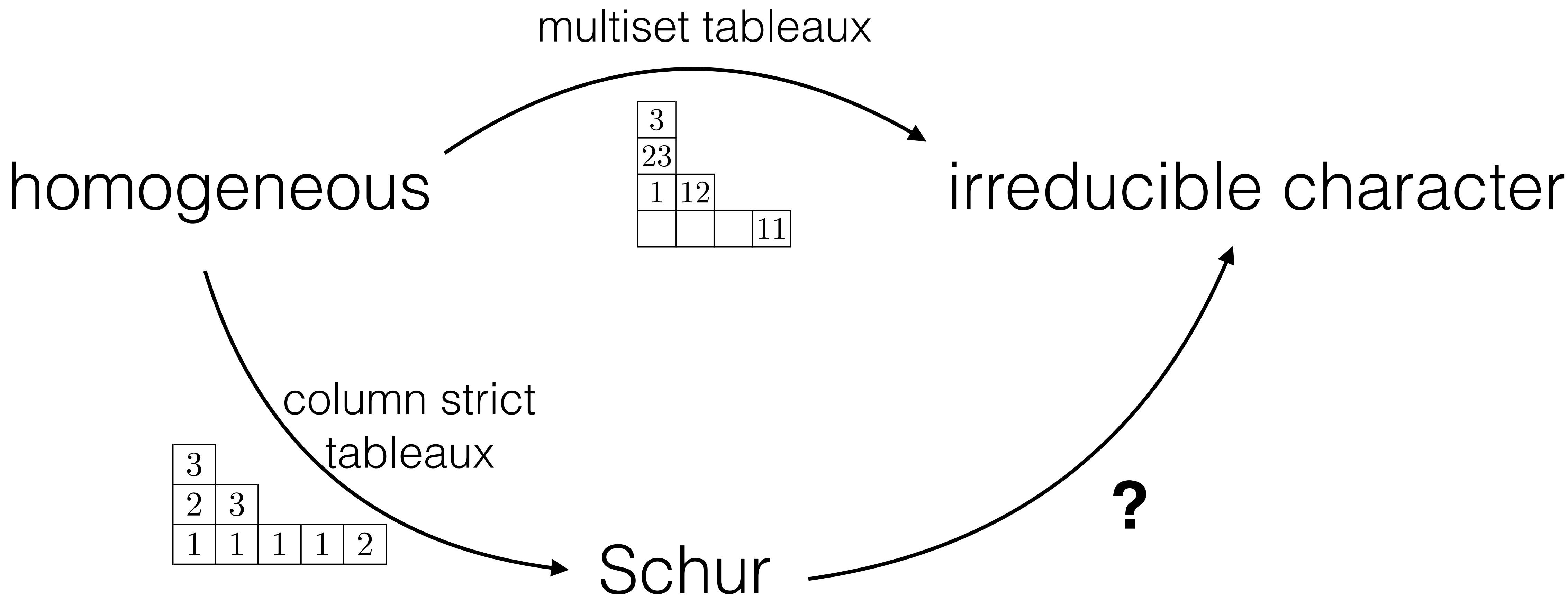
The symmetric group realized as permutation matrices sits inside of the general linear group

Q: How does an irreducible Gln representation decompose as an Sn irreducible representation?

A: translated in terms of characters: expand a Schur function terms of the irreducible character basis


multiset tableaux
homogeneous
irreducible character
column strict
tableaux
Schur
?



column strict tableaux $\times$ **?** $\longleftrightarrow$ multiset tableaux

```
sage: Sym = SymmetricFunctions(QQ)
sage: s = Sym.schur()
sage: st = Sym.irreducible_symmetric_group_character()
```

```
sage: st[1]*st[2,1]
st[1, 1] + st[1, 1, 1] + st[2] + 2*st[2, 1] + st[2, 1, 1] + st[2, 2] +
st[3] + st[3, 1]
```

```
sage: st(s[2,1])
st[] + 3*st[1] + 2*st[1, 1] + 2*st[2] + st[2, 1]
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“Sage is the best thing out there for doing symmetric functions”

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$$(f+g)[h] = f[h] + g[h]$$

$$(f \cdot g)[h] = f[h]g[h]$$

$$p_k[f+g] = p_k[f] + p_k[g]$$

$$p_k[p_r] = p_{kr}$$

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“outer” plethysm

$$s_\lambda[s_\mu] = \sum_\nu g^\nu_{\lambda\mu} s_\nu$$

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sage: s = Sym.schur()

sage: s[2,1](s[2,2])
```

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s[3, 2, 2, 2, 2, 1] + s[3, 3, 2, 2, 1, 1] + s[3, 3, 3, 2, 1] + s[4, 2, 2, 2, 2] + s[4, 3, 2, 1,
1, 1] + 2*s[4, 3, 2, 2, 1] + s[4, 3, 3, 1, 1] + s[4, 3, 3, 2] + s[4, 4, 2, 1, 1] + 2*s[4, 4, 2,
2] + s[4, 4, 3, 1] + s[5, 2, 2, 2, 1] + s[5, 3, 2, 1, 1] + s[5, 3, 2, 2] + s[5, 3, 3, 1] + s[5,
4, 1, 1, 1] + 2*s[5, 4, 2, 1] + s[5, 4, 3] + s[5, 5, 1, 1] + s[6, 3, 2, 1] + s[6, 4, 2] + s[6,
5, 1]
```

Lesson learned in preparing this talk

```
sage: st[1,1](s[2])
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what to do?

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sage: s = Sym.schur()
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5, 1]
```

Open problem #4:

Q: How does the composition of Gln and Sn representation decompose as Sn representations?

```
sage: st = Sym.irreducible_symmetric_group_character()
sage: st(s[2,1](st[2]))
```

```
2*st[1] + 4*st[1, 1] + 3*st[1, 1, 1] + st[1, 1, 1, 1] + 5*st[2] + 9*st[2, 1] + 5*st[2, 1, 1] +
st[2, 1, 1, 1] + 5*st[2, 2] + 2*st[2, 2, 1] + 4*st[3] + 7*st[3, 1] + 3*st[3, 1, 1] + 3*st[3, 2]
+ st[3, 2, 1] + 3*st[4] + 3*st[4, 1] + st[4, 2] + st[5] + st[5, 1]
```

"inner" plethysm

$$s_\lambda[\tilde{s}_\mu] = \sum_\nu d^\nu_{\lambda\mu} \tilde{s}_\nu$$