Pieri rules and combinatorics of symmetric group characters

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For each $\lambda \vdash k$ there is a polynom[i]
irred representation of Gl_n of dim
m where $m = \#$ CST of shape λ and
content $\{1, 2, \dots, n\}$

$$\lambda = (2,1)$$

Example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} a^2d - acb & b^2c - abd \\ bc^2 - acd & ad^2 - abc \end{bmatrix}$$

$$v_{\begin{smallmatrix} 2 & \\ 1 & 1 \end{smallmatrix}} = x_1x_1x_2 - x_1x_2x_1$$

$$v_{\begin{smallmatrix} 2 & \\ 1 & 2 \end{smallmatrix}} = x_2x_2x_1 - x_2x_1x_2$$

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$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \longrightarrow \begin{bmatrix} \text{some } 8 \times 8 \\ \text{matrix} \end{bmatrix}$$

$$\begin{array}{|c|c} \hline 2 \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 3 \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 2 \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 3 \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 2 \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 3 \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 3 \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 3 \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

For each $\lambda \vdash k$ there is a polynom irred representation of Gl_n of dim m where $m = \#$ CST of shape λ and content $\{1, 2, \ldots, n\}$

Example:

$$\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1^2 x_2 & 0 \\ 0 & x_1 x_2^2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1^2x_2 & & & & & & & \\ & x_1^2x_3 & & & & & & \\ & & x_1x_2^2 & & & & & \\ & & & x_1x_2x_3 & & & & \\ & & & & x_1x_2x_3 & & & \\ & & & & & x_1x_3^2 & & \\ & & & & & & x_2^2x_3 & \\ & & & & & & & x_2x_3^2 \end{bmatrix}$$

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Example:

$$\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1^2 x_2 & 0 \\ 0 & x_1 x_2^2 \end{bmatrix}$$

$$s_{21}(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2$$

$$\begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1^2x_2 & & & & & & & \\ & x_1^2x_3 & & & & & & \\ & & x_1x_2^2 & & & & & \\ & & & x_1x_2x_3 & & & & \\ & & & & x_1x_2x_3 & & & \\ & & & & & x_1x_3^2 & & \\ & & & & & & x_2^2x_3 & \\ & & & & & & & x_2x_3^2 \end{bmatrix}$$

$$s_{21}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

GL_n

$s_\lambda(x_1, x_2, \dots, x_n)$

$$Gl_n \supset O_n$$

$$s_\lambda(x_1, x_2, \ldots, x_n) \qquad o_\lambda(x_1, x_2, \ldots, x_n)$$

$$Gl_n \supset O_n \supset S_n$$

$s_\lambda(x_1, x_2, \ldots, x_n)$ $\quad$ $o_\lambda(x_1, x_2, \ldots, x_n)$ $\quad$ $\tilde{s}_\lambda(x_1, x_2, \ldots, x_n)$

σ a permutation matrix $n \times n$

cycle type $\sigma = \mu$ $\quad$ eigenvals Ξ_μ

$$\tilde{s}_\lambda(\Xi_\mu) = \chi^{(n-|\lambda|,\lambda)}(\mu)$$

Example

$$\tilde{S}_1(x_1, x_2, x_3) = x_1 + x_2 + x_3 - 1$$

group element	$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
eigenvalues	$1, 1, 1$	$1, -1, 1$	$1, \zeta, \zeta^2$
$\tilde{S}_1(x_1, x_2, x_3)$	2	0	-1

character table for S_3

	C_1	C_2	C_3
$\zeta^{(1)}$	1	1	1
$\zeta^{(2)}$	1	-1	1
$\zeta^{(3)}$	2	0	-1

$$GL_n \supset O_n \supset S_n$$

$$s_\lambda(x_1, x_2, \ldots, x_n) \qquad o_\lambda(x_1, x_2, \ldots, x_n) \qquad \tilde{s}_\lambda(x_1, x_2, \ldots, x_n)$$

$$Gl_n \supset O_n \supset S_n$$

$$s_\lambda(x_1, x_2, \ldots, x_n) \qquad o_\lambda(x_1, x_2, \ldots, x_n) \qquad \tilde{s}_\lambda(x_1, x_2, \ldots, x_n)$$

$$h_\mu = \sum_\lambda M_{\lambda\mu} \tilde{s}_\lambda$$

$M_{\lambda\mu} =$ # of multi-set tableaux of shape $(r, \lambda)/(\lambda_1)$ content μ

24	34	4				
234	3	34				
13	13	22	23			
1	112	12	12			
				11	111	2

$\mu = (12, 9, 7, 5)$

$\lambda = (4, 4, 3, 3)$

$$Gl_n \supset O_n \supset S_n$$

$s_\lambda(x_1, x_2, \ldots, x_n)$ $\quad$ $o_\lambda(x_1, x_2, \ldots, x_n)$ $\quad$ $\tilde{s}_\lambda(x_1, x_2, \ldots, x_n)$

$$\boxed{h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda}$$

$$\boxed{h_\mu = \sum_{\lambda,\gamma,\nu} K_{\lambda\mu} c^{\lambda}_{2\gamma,\nu} o_\nu}$$

$$\boxed{h_\mu = \sum_\lambda M_{\lambda\mu} \tilde{s}_\lambda}$$

structure coefficients

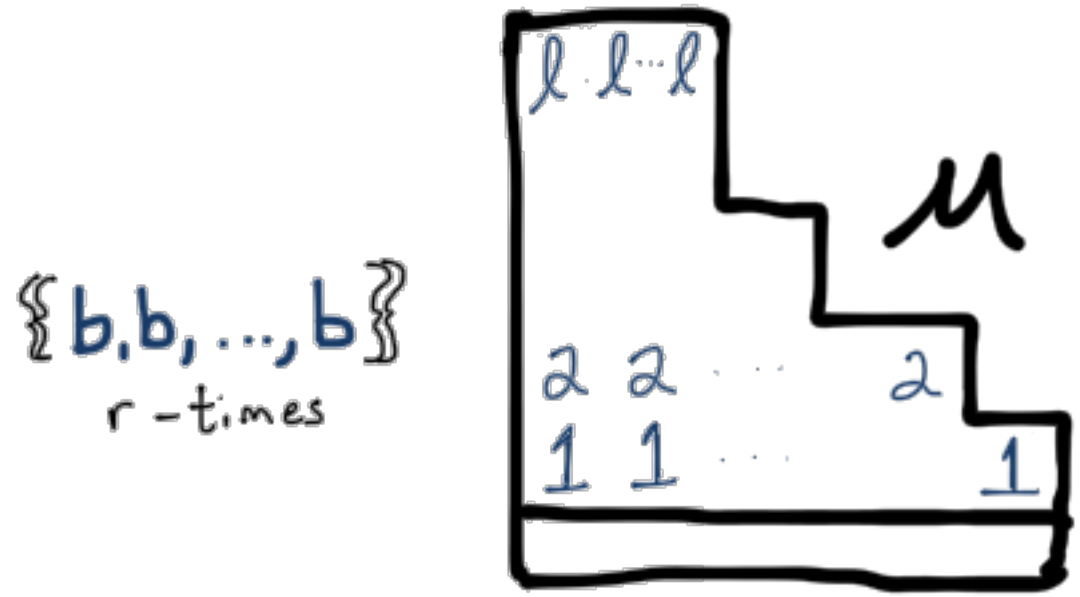
$$\tilde{S}_\lambda \tilde{S}_\mu = \sum_\nu \bar{g}_{\lambda\mu\nu} \tilde{S}_\nu$$

reduced Kronecker coefficients of Schur functions

$$S_{(n-|\lambda|,\lambda)} * S_{(n-|\mu|,\mu)} = \sum_\nu \bar{g}_{\lambda\mu\nu} S_{(n-|\nu|,\nu)}$$

Pieri rule

$$h_r \tilde{S}_\mu$$



1. cells are labelled $\{\!\{j\}\!\}$ $\{\!\{b^i\}\!\}$ or $\{\!\{j, b^i\}\!\}$

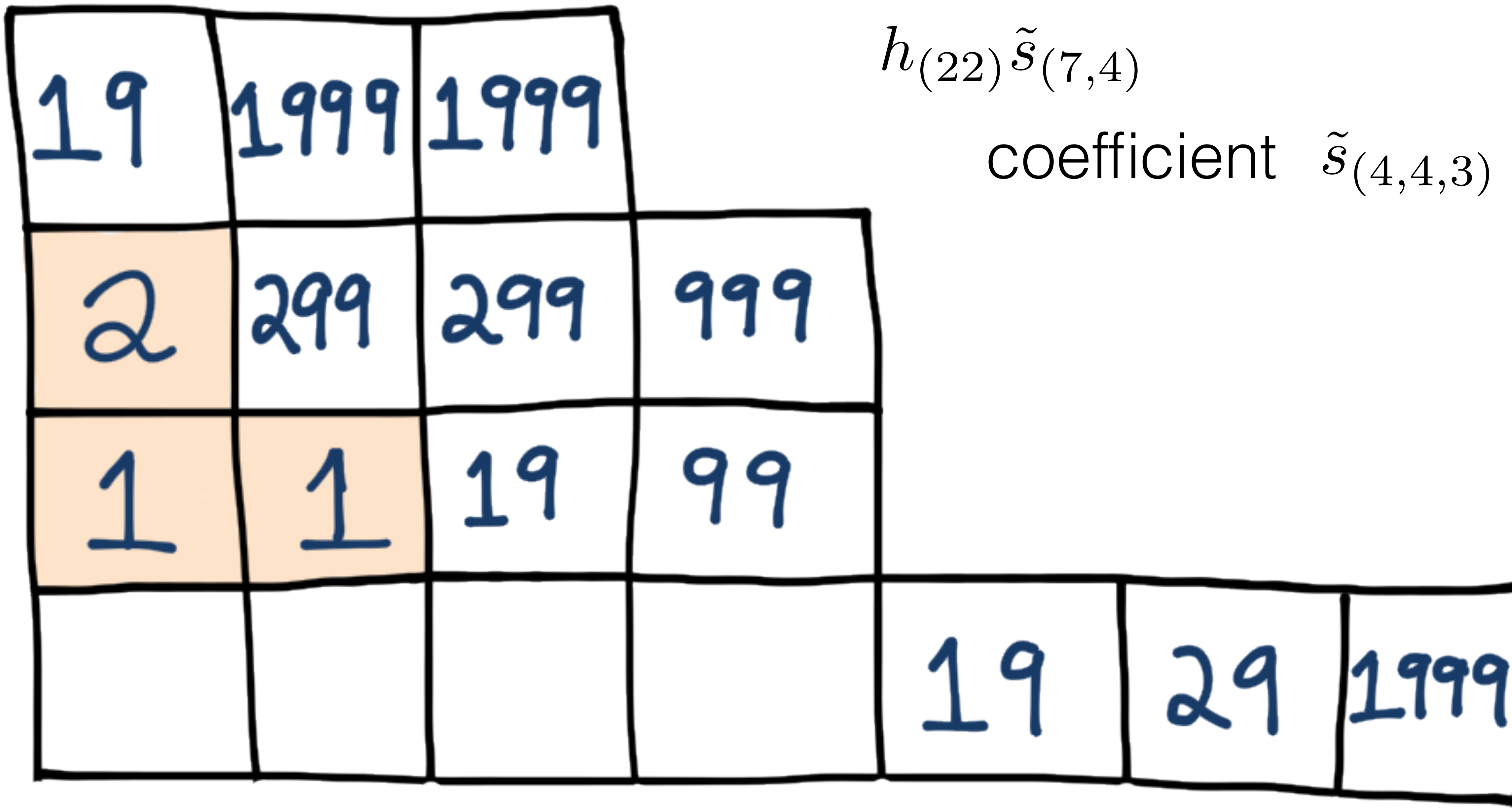
2. column strict + cells labelled with $\{\!\{j\}\!\}$
   stay in their place

3. there is a ‘lattice’ condition

19	1999	1999				
2	299	299	999			
1	1	19	99			
				19	29	1999

$h_{(22)}\tilde{s}_{(7,4)}$

coefficient $\tilde{s}_{(4,4,3)}$



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coefficient $\tilde{s}_{(4,4,3)}$

211

19	1999	1999				
2	299	299	999			
1	1	19	99			
				19	29	1999

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1112 211

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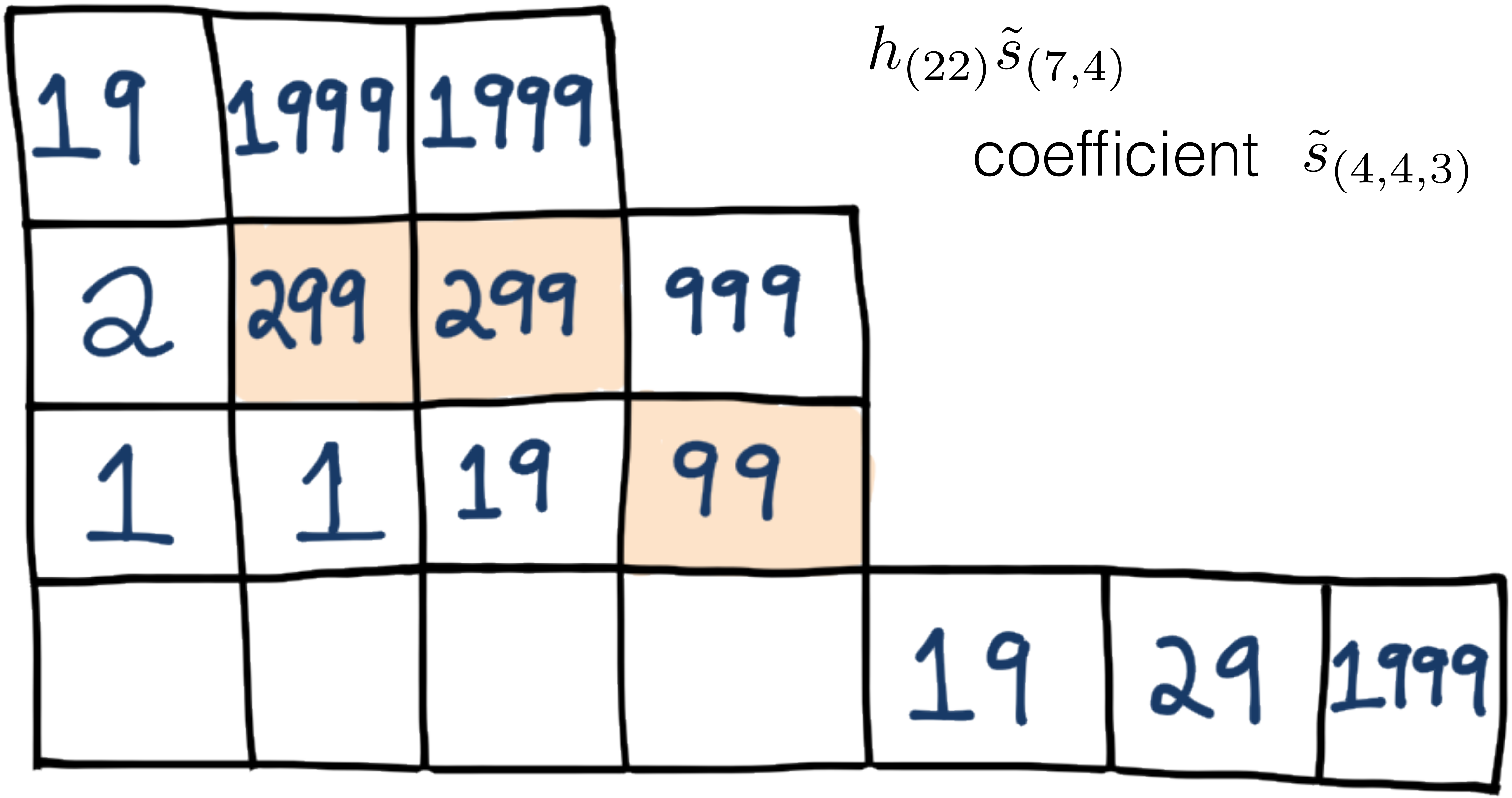

22 1112 211

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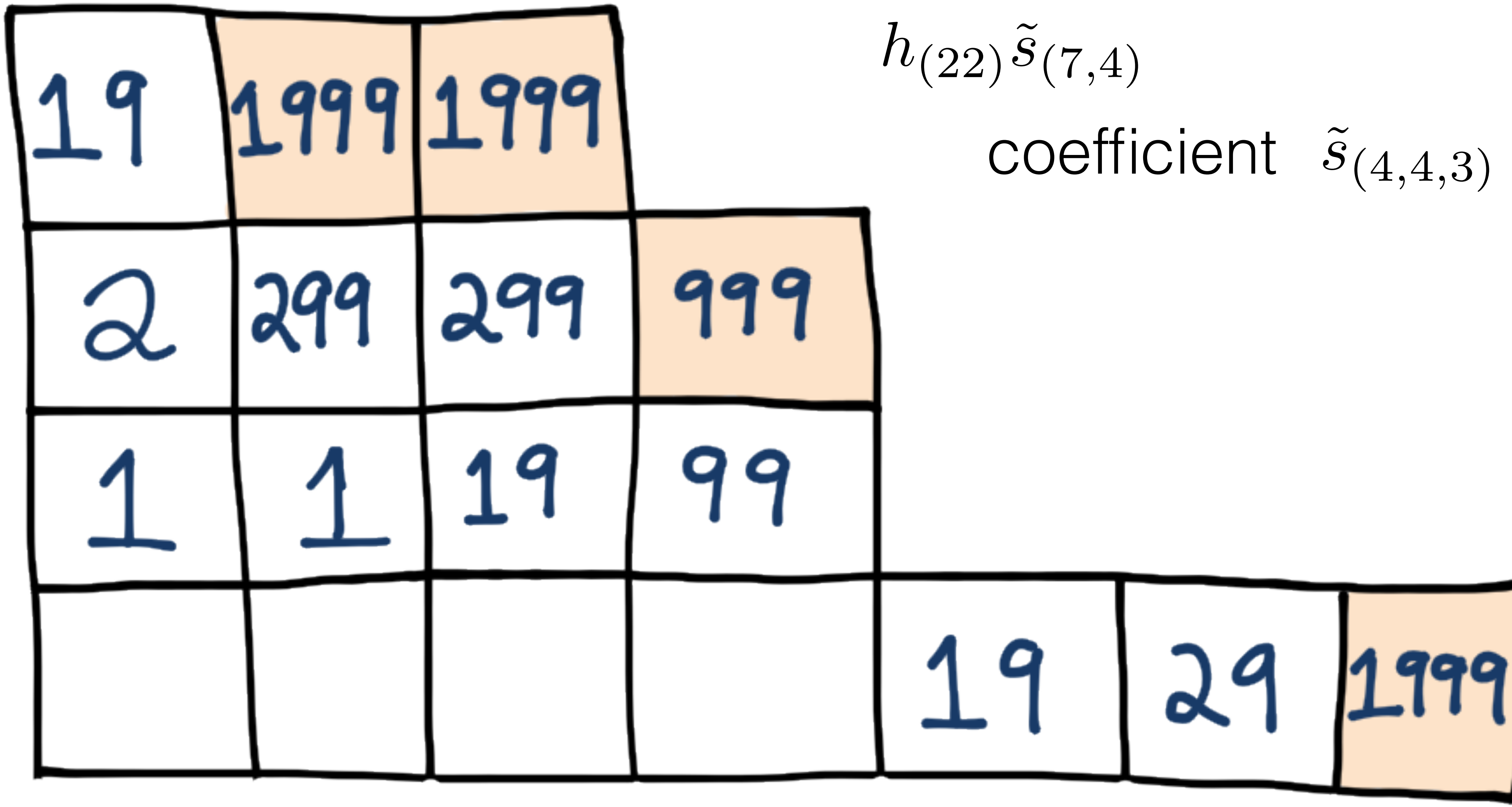

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Thank you!

Frobenius map

$$\phi_n(f) = \frac{1}{n!} \sum_{\sigma \in S_n} f(\text{eigenvalues}(\sigma))\, p_{\lambda(\sigma)}$$

Image of irreducible character basis

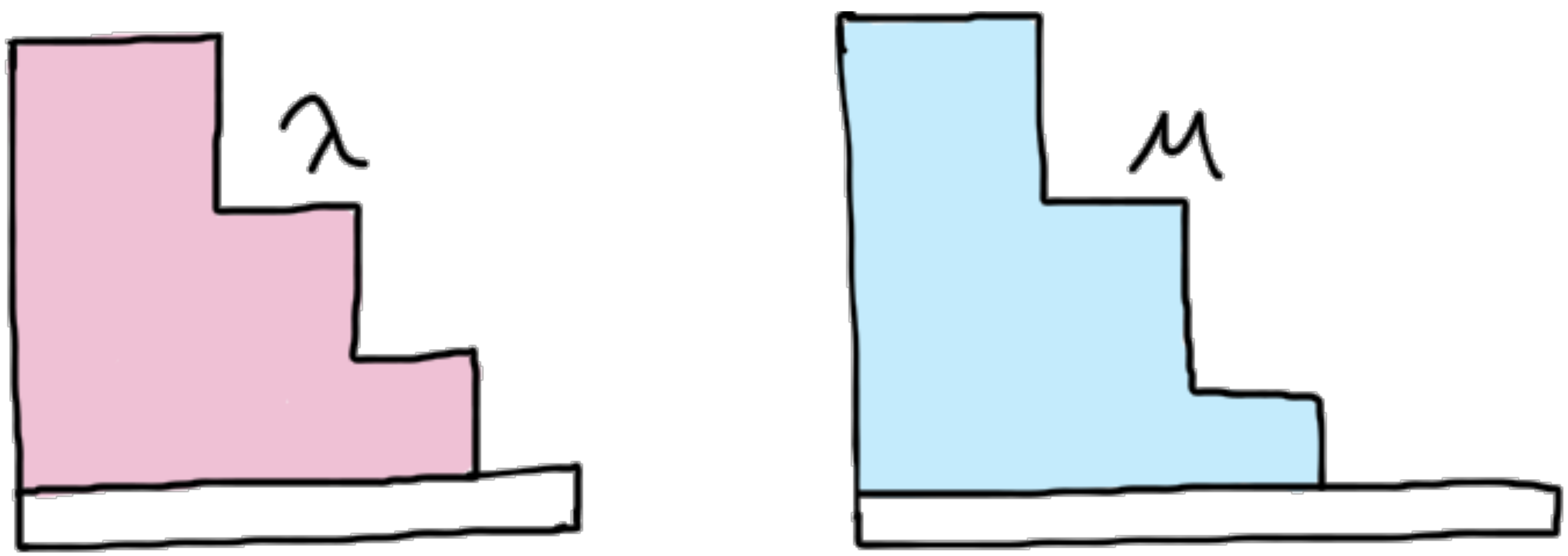
$$\phi_n(\tilde{S}_\lambda) = S_{(n-|\lambda|, \lambda)}$$

Scalar product

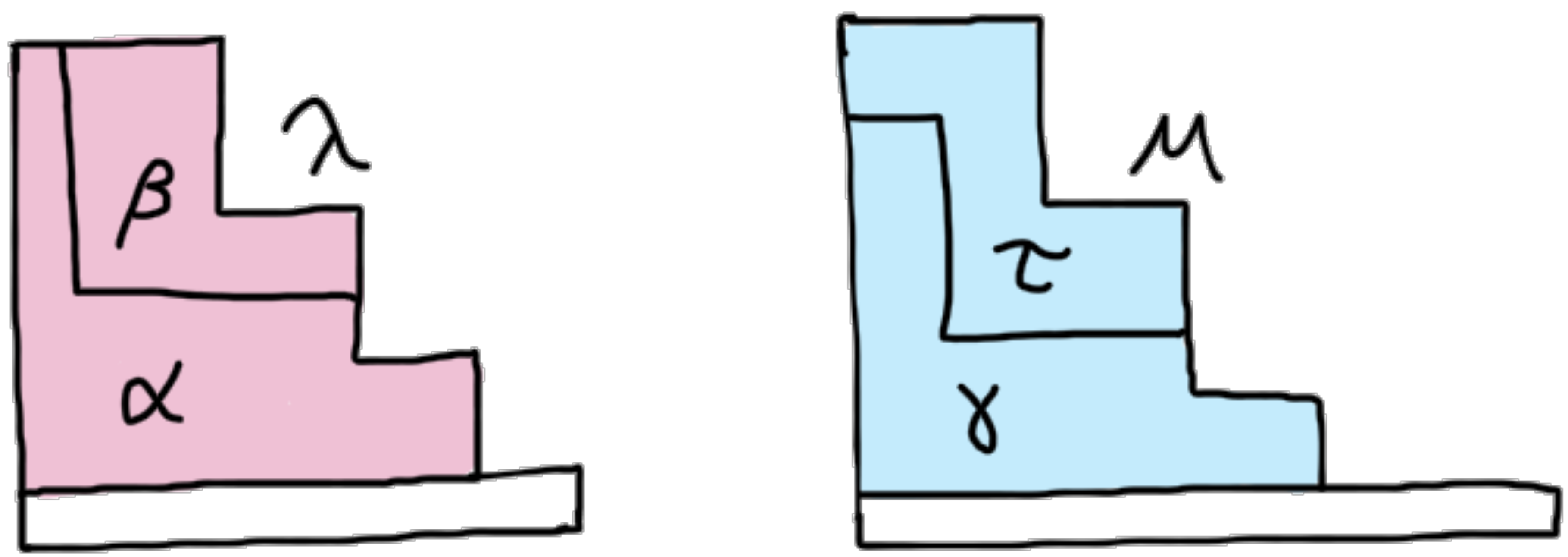
$$\langle f, g \rangle_{@} := \langle \phi_n(f), \phi_n(g) \rangle$$

$$\langle \tilde{S}_\lambda, \tilde{S}_\mu \rangle_{@} = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{else} \end{cases}$$

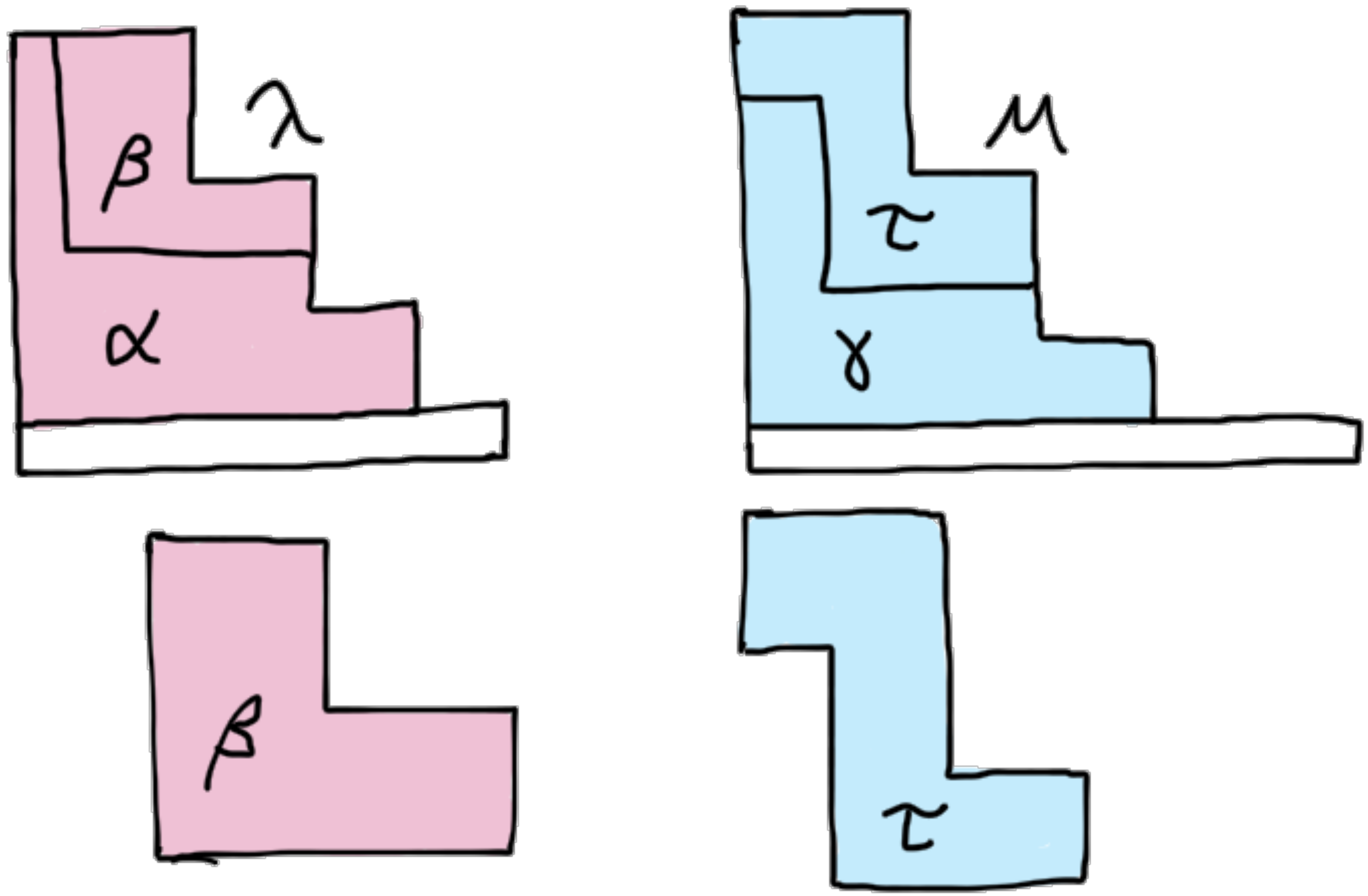
structure coefficients again



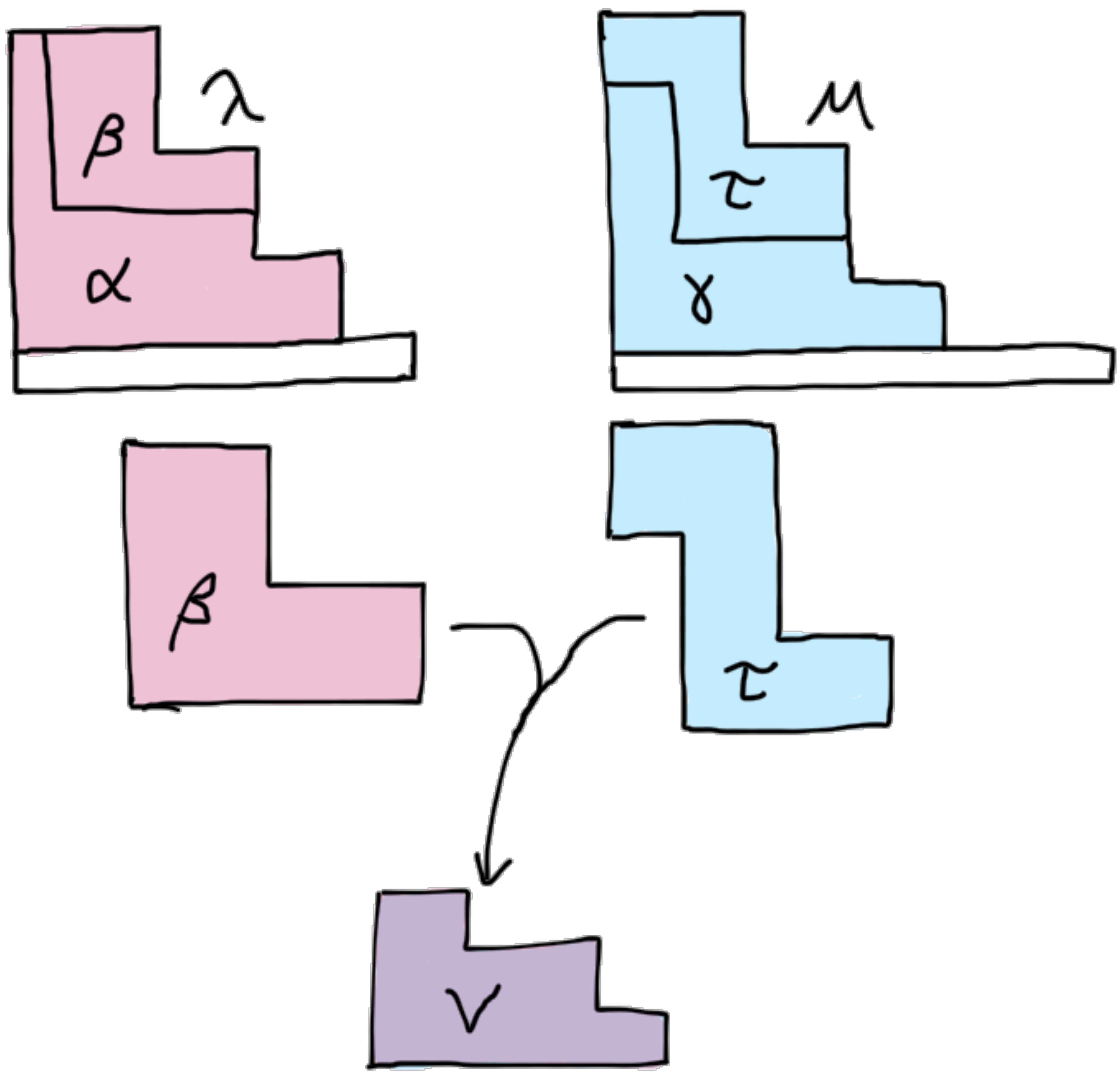
structure coefficients again



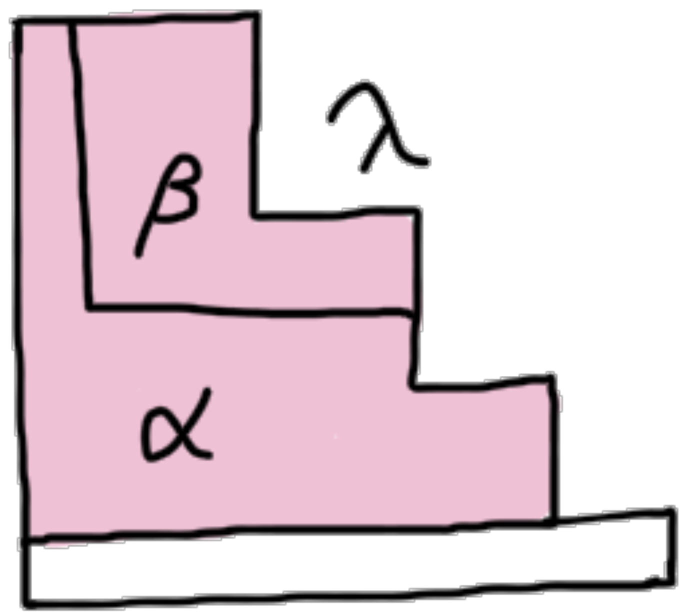
structure coefficients again

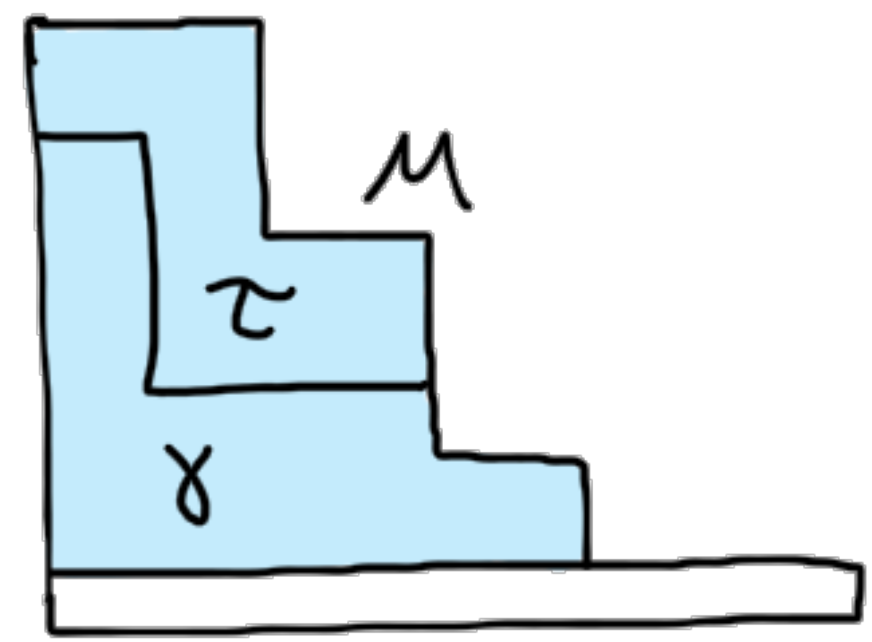


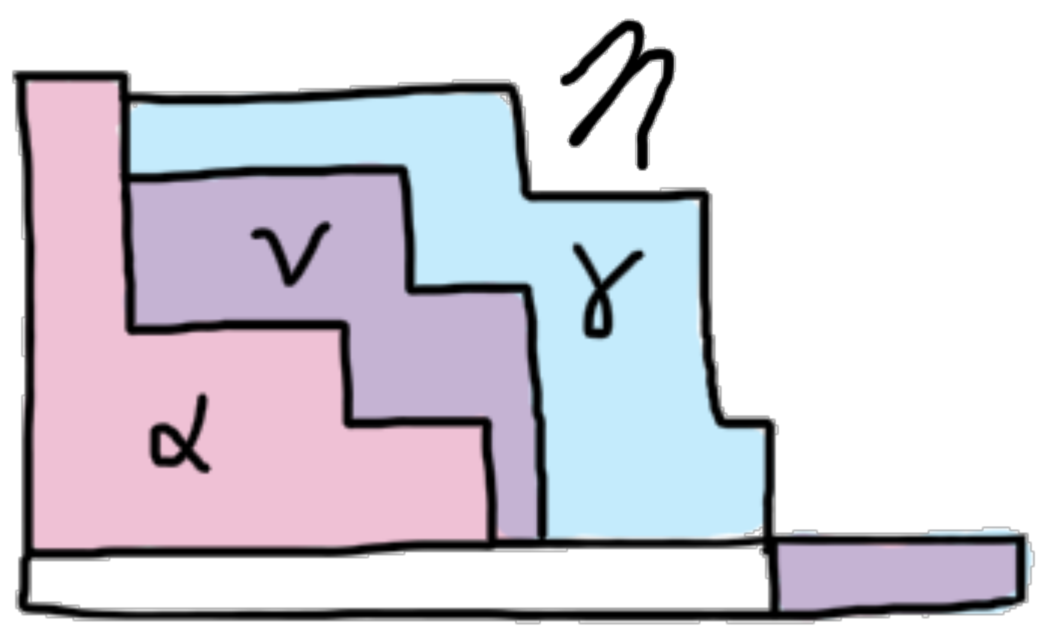
structure coefficients again



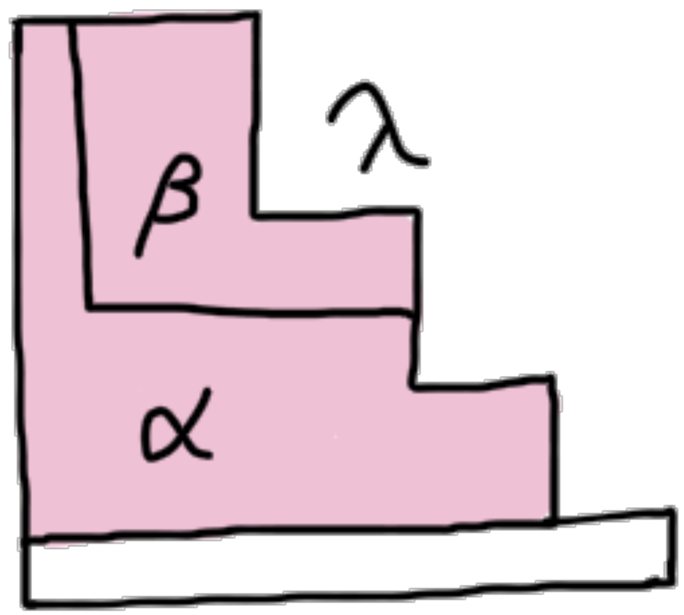
structure coefficients again

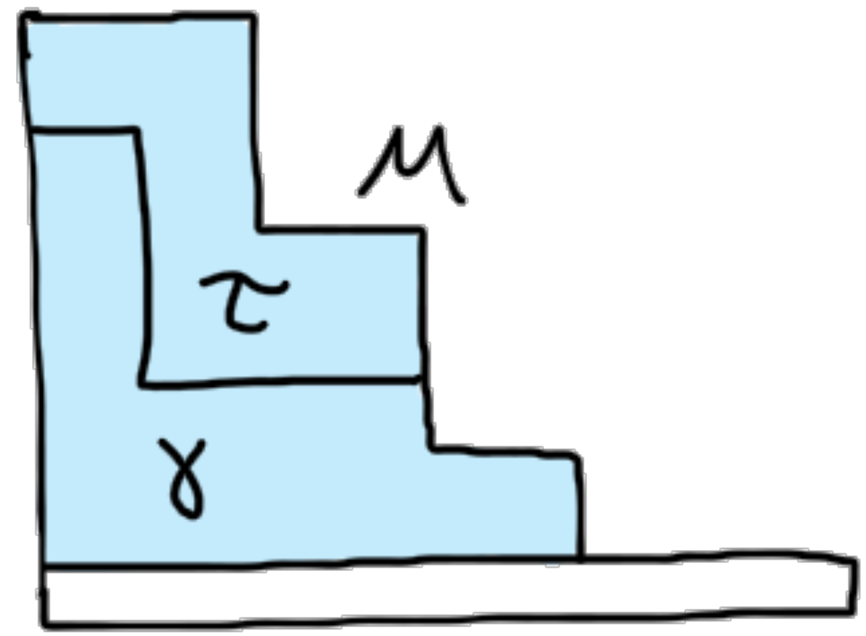


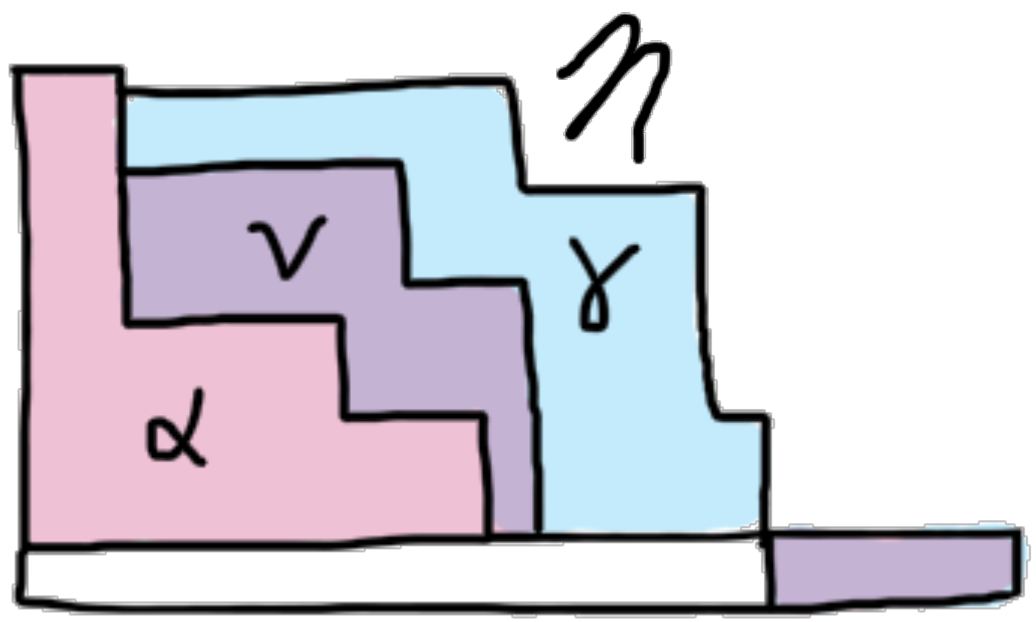




structure coefficients again







$$\overline{g}_{\lambda\mu\eta} = \sum_{\alpha,\beta} \sum_{\gamma,\tau} \sum_{\sigma} c^{\lambda}_{\alpha\beta} c^{\mu}_{\gamma\tau} g_{\beta\tau\nu} c^{\nu}_{\sigma(|\tau|-|\sigma|)} c^{\eta}_{\lambda\sigma\mu}$$