Combinatorics of characters of symmetric group as symmetric functions

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joint work with Rosa Orellana

The ring of symmetric functions' dual role in representation theory

Sym_{X_n} is the ring of characters of $Gl_n(\mathbb{C})$

$$s_\lambda(x_1, x_2, \ldots, x_n)$$

Sym is isomorphic to the ring of characters of $\bigoplus_{k \geq 0} S_k$

$$\mathcal{F}_{S_k}(\chi^\lambda) = s_\lambda$$

λ partition of k and χ^λ is an irreducible S_k character

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$$\cup|$$

Sym_{X_n} is the ring of characters of S_n

$$\tilde{s}_\lambda(x_1, x_2, \ldots, x_n)$$

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$$\tilde{s}_\lambda(x_1, x_2, \ldots, x_n)$$

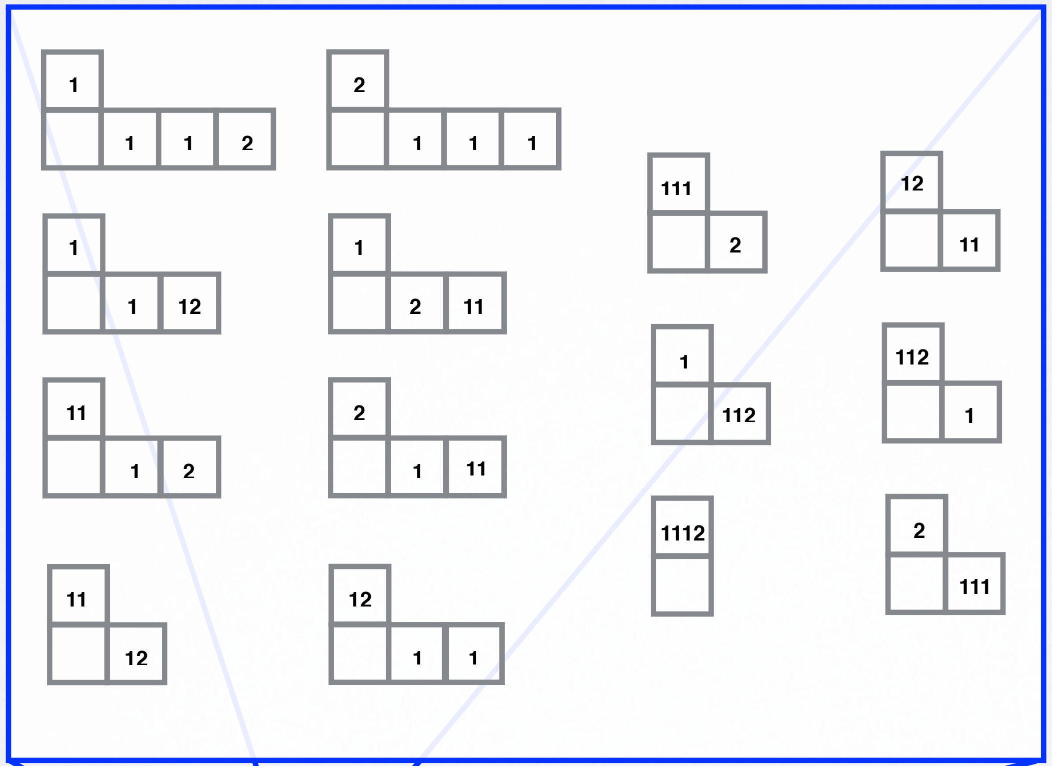
$$\tilde{s}_\lambda[\text{eigenvals of permutation matrix } \mu] = \chi^{(n-|\lambda|,\lambda)}(\mu)$$

$$\tilde{s}_\lambda \tilde{s}_\nu = \sum_\gamma \overline{k}_{\lambda\nu\gamma} \tilde{s}_\gamma$$

Theorem

The coefficient of $\tilde{s}_\lambda$ in h_μ

is the number of column strict tableaux of shape (r, λ) and content μ whose entries are multi-sets



$$h_{31} = 7\tilde{s}_{()} + 14\tilde{s}_1 + 8\tilde{s}_{11} + \tilde{s}_{111} + 10\tilde{s}_2$$
$$+4\tilde{s}_{21} + 4\tilde{s}_3 + \tilde{s}_{31} + \tilde{s}_4$$

Discovered and rediscovered….

Littlewood 1958

The characters of the symmetric group can be obtained from those of the full linear group in a similar manner to that used for the orthogonal group, namely by considering a tensor corresponding to any partition (λ) of any integer n, and removing all possible contractions with the fundamental forms (**2**, p. 392). The remainder when all contractions are removed is an irreducible character, provided that $n - p \geqslant \lambda_1$, and it is not difficult to see that it is in fact the character of the symmetric group corresponding to the partition $(n - p, \lambda_1, \ldots, \lambda_i)$. It is convenient to represent by $[\lambda]$ not this character, but the corresponding S-function

$$[\lambda] = \{n - p, \lambda_1, \ldots, \lambda_i\}.$$

$$[21]\cdot[21] = [42] + [41^2] + [3^2] + 2[321] + [2^3] + [31^3] + [2^21^2] + [5] + 4[41] + 5[32] + 6[31^2] + 5[2^21] + 4[21^3] + [1^5] + 3[4] + 9[31] + 6[2^2] + 9[21^2] + 3[1^4] + 5[3] + 9[21] + 5[1^3] + 4[2] + 4[1^2] + 2[1] + [0].$$

Butler, King 1973

The symmetric groups are thus treated quite differently from the linear and other continuous groups: the orthogonal, rotation, and symplectic groups. The characters of these groups are known[8] in terms of S functions and the usual method of calculating such things as Kronecker products of the representations of these groups is to use S-functional expressions for their characters and the powerful algebra of S functions associated with the n-independent outer product rule. The labels that arise from this approach are the same as those that arise from tensorial arguments.[7,9] The aim of this paper is to show that the symmetric groups, Σ_n, may be treated in an n-independent manner similar to that used for the restricted groups O_n and Sp_n, rather than in the usual n-dependent manner requiring a development of the somewhat complicated algebra of inner products of S functions.[10]

Some specific examples of (3.4) are of interest, namely:

$$L_{n-1} \to \Sigma_n \; \{1\} \to \langle 1\rangle \qquad (3.6a)$$

$$\{2\} \to \langle 2\rangle + \langle 1\rangle + \langle 0\rangle \qquad (3.6b)$$

$$\{1^2\} \to \langle 1^2\rangle \qquad (3.6c)$$

$$\{1^k\} \to \langle 1^k\rangle \qquad (3.6d)$$

$$\{1^{n-1}\} \to \langle 1^{n-1}\rangle. \qquad (3.6e)$$

Speyer 2010

mathoverflow

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Reference request: The stable Kronecker ring is isomorphic to the ring of symmetric polynomials

asked 7 years, 7 months ago
viewed 451 times
active 5 days ago

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Background

For λ any partition and n a positive integer, write $\lambda[n]$ for the sequence $(n - |\lambda|, \lambda_1, \lambda_2, \ldots, \lambda_r)$. For n large enough, this is a partition of n.

The irreducible representations of S_n are indexed by partitions of n; we denote them by S_λ. The Kronecker coefficients $g^{\nu}_{\lambda\mu}$ are defined by the equality

$$S_\lambda \otimes S_\mu \cong \bigoplus g^{\nu}_{\lambda\mu} S_\nu$$

Question

I can prove that the stable Kronecker ring is isomorphic to the ring of symmetric functions. Is this fact already in the literature?

co.combinatorics | reference-request | symmetric-group

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asked Jan 4 '10 at 21:21
David Speyer
93.4k • 7 • 221 • 464

Why? Applications

Church-Farb representation stability

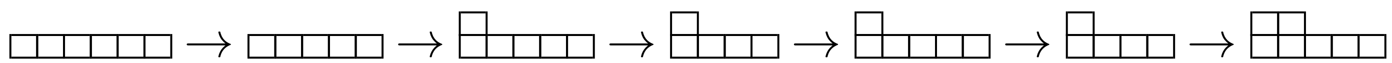
representation theory of symmetric group and the partition algebra

Kronecker and reduced/stable Kronecker product

restriction/branching from irreducible Gln to Sn

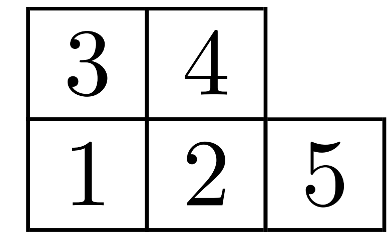
plethysm

Combinatorics of multi-set partitions, multi-set tableaux and Hopf algebras



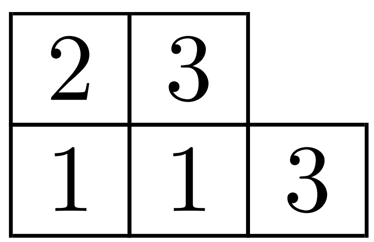
partition algebra irrep dimensions

oscillating tableaux



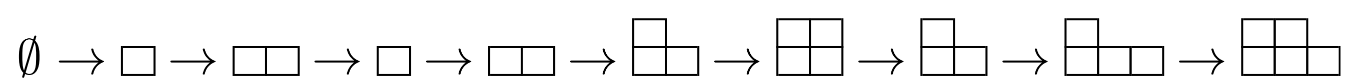
symmetric group irrep dimensions

standard tableaux



general linear group irrep dimensions

column strict tableaux



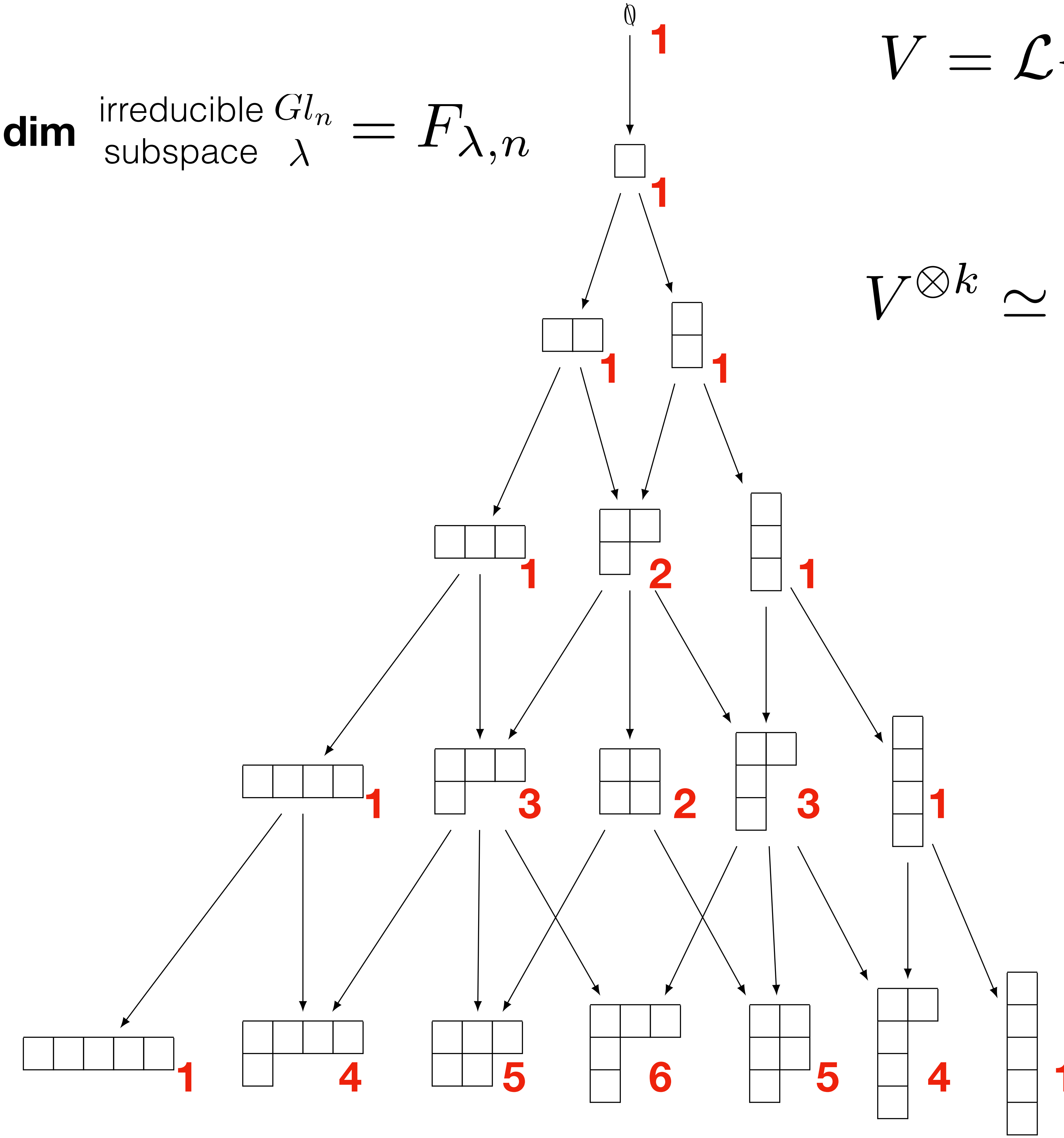
Brauer algebra irrep dimensions

vascillating tableaux

$$V = \mathcal{L}\{v_1, v_2, \ldots, v_n\}$$

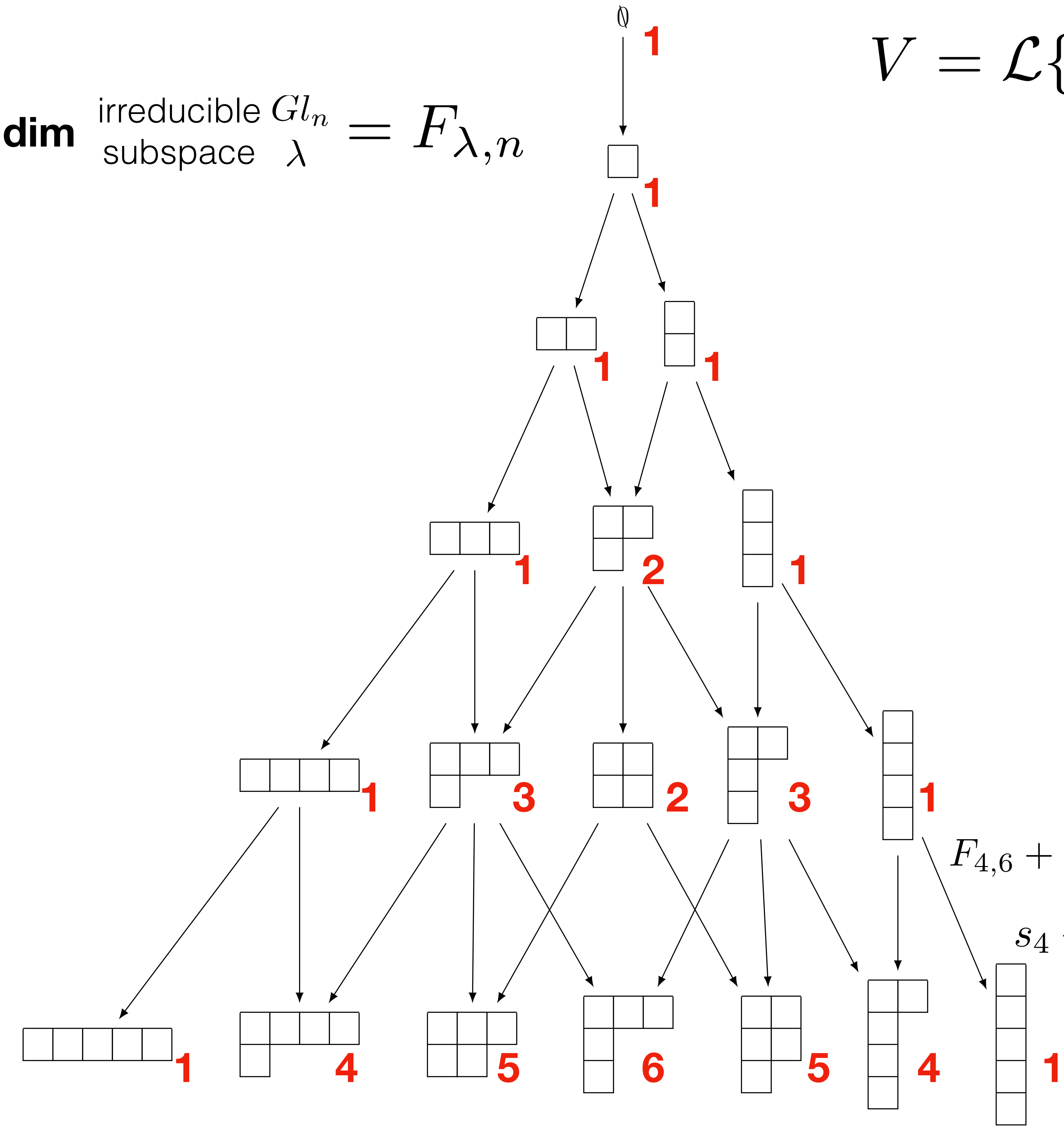
$$\textbf{dim} \begin{pmatrix}\text{irreducible } Gl_n \\ \text{subspace } \lambda\end{pmatrix} = F_{\lambda,n}$$

$$V^{\otimes k} \simeq \bigoplus_{\lambda \vdash k} \begin{pmatrix}\text{irreducible } Gl_n \\ \text{subspace } \lambda\end{pmatrix}^{\oplus f_\lambda}$$



dim irreducible Gl_n subspace λ $= F_{\lambda,n}$

$$V = \mathcal{L}\{v_1, v_2, \ldots, v_n\}$$



dimension	$F_{1,6} = 6$
character	$s_1 = h_1$

$$F_{2,6} + F_{11,6} = 36$$

$$s_2 + s_{11} = h_{11}$$

$$F_{3,6} + 2F_{21,6} + F_{111,6} = 216$$

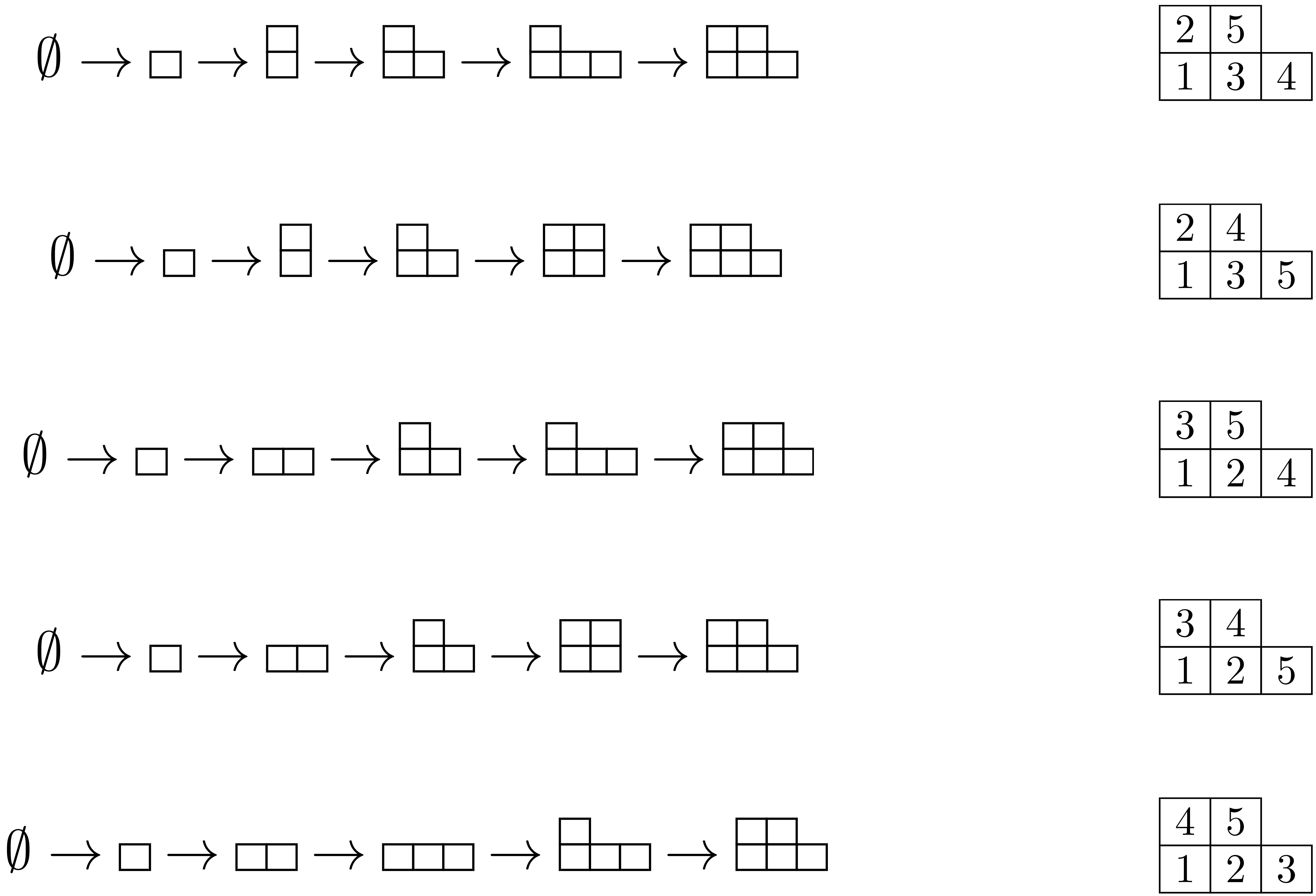
$$s_3 + 2s_{21} + s_{111} = h_{111}$$

$$F_{4,6} + 3F_{31,6} + 2F_{22,6} + 3F_{211,6} + F_{1111,6} = 6^4$$

$$s_4 + 3s_{31} + 2s_{22} + 3s_{211} + s_{111} = h_{1111}$$

$$F_{5,6} + 4F_{41,6} + 5F_{32,6} + 6F_{311,6} + 5F_{221,6} + 4F_{2111,6} + F_{1^5,6} = 6^5$$

$$s_5 + 4s_{41} + 5s_{32} + 6s_{311} + 5s_{221} + 4s_{2111} + s_{1^5} = h_{1^5}$$


∅
2 5
1 3 4
∅
2 4
1 3 5
∅
3 5
1 2 4
∅
3 4
1 2 5
∅
4 5
1 2 3

$$V = \mathcal{L}\{v_1, v_2, \ldots, v_n\}$$

$$V^{\otimes k} \simeq \bigoplus_{\lambda \vdash k} \begin{pmatrix} \text{irreducible} & Gl_n \\ \text{subspace} & \lambda \end{pmatrix}^{\oplus f_\lambda}$$

$$n^k = \sum_{\lambda \vdash k} \begin{pmatrix} \text{\# of column} \\ \text{strict tableaux} \\ \text{of shape } \lambda \end{pmatrix} \begin{pmatrix} \text{\# of standard} \\ \text{tableaux of} \\ \text{shape } \lambda \end{pmatrix}$$

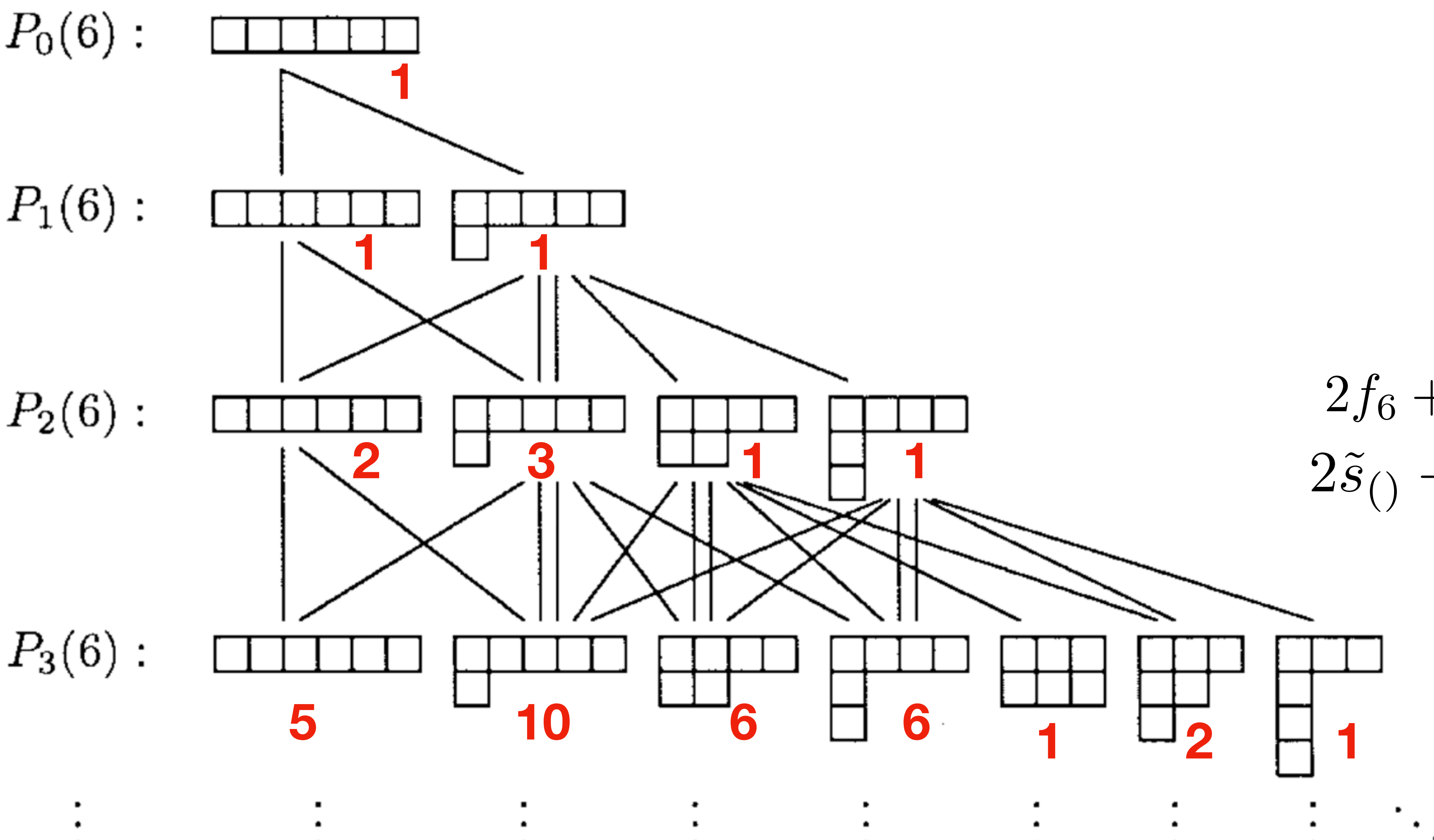
$$h_{1^k}(x_1, x_2, \ldots, x_n) = \sum_{\lambda \vdash k} f_\lambda s_\lambda(x_1, x_2, \ldots, x_n)$$

$$V = \mathcal{L}\{v_1, v_2, \ldots, v_n\}$$

$$V^{\otimes k} \simeq \bigoplus_{\lambda:|\lambda|\leq k} \begin{pmatrix} \text{irreducible } & S_n \\ \text{subspace} & \lambda \end{pmatrix}^{\oplus n_\lambda}$$

$$n^k = \sum_{\lambda \vdash n} \begin{pmatrix} \text{\# of standard} \\ \text{tableaux shape} \\ \lambda \end{pmatrix} \begin{pmatrix} \text{paths in a Bratteli} \\ \text{diagram} \end{pmatrix}$$

$$h_{1^k}(x_1, x_2, \ldots, x_n) = \sum_{\lambda:|\lambda|\leq k} n_\lambda \tilde{s}_\lambda(x_1, x_2, \ldots, x_n)$$

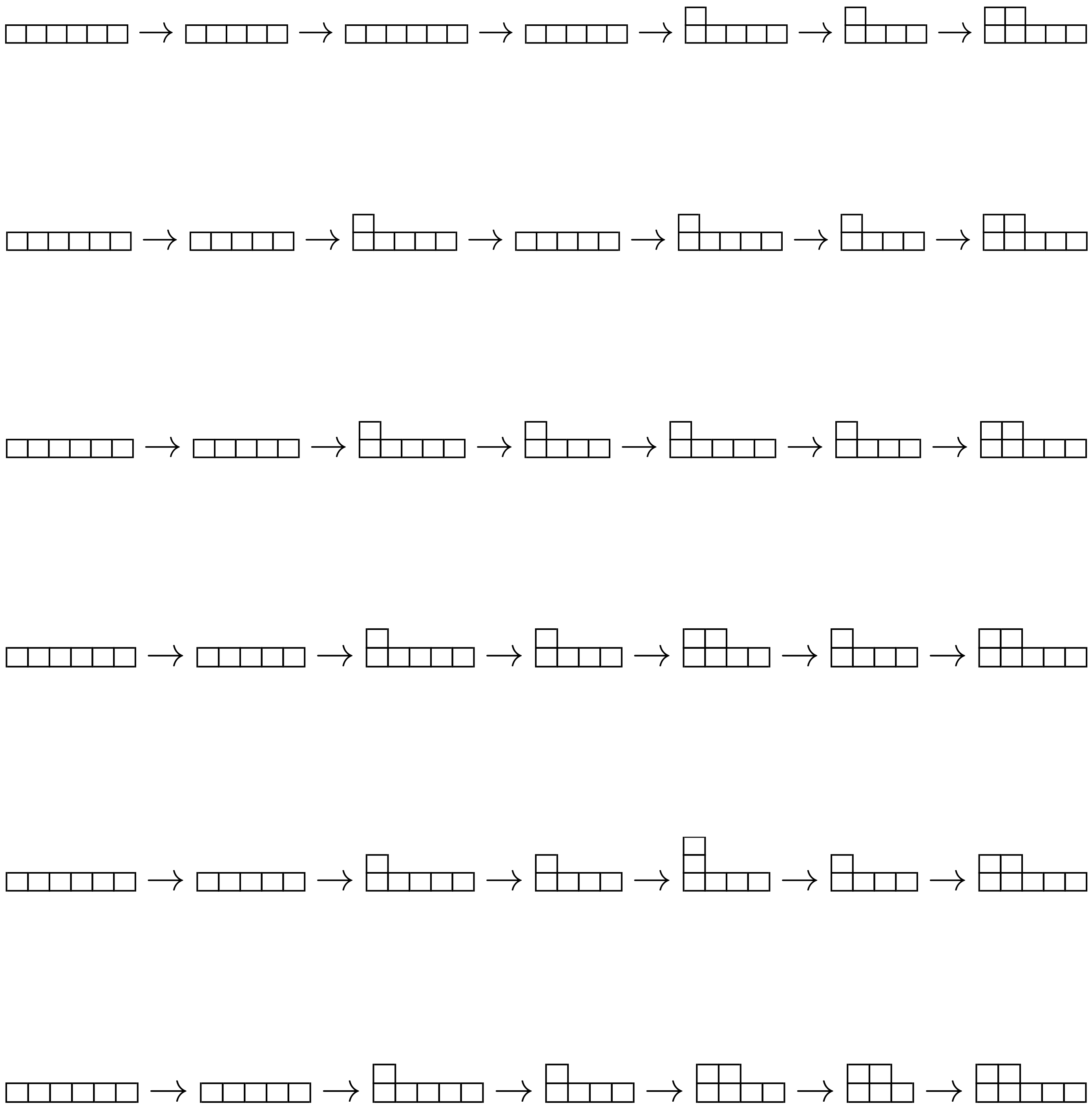


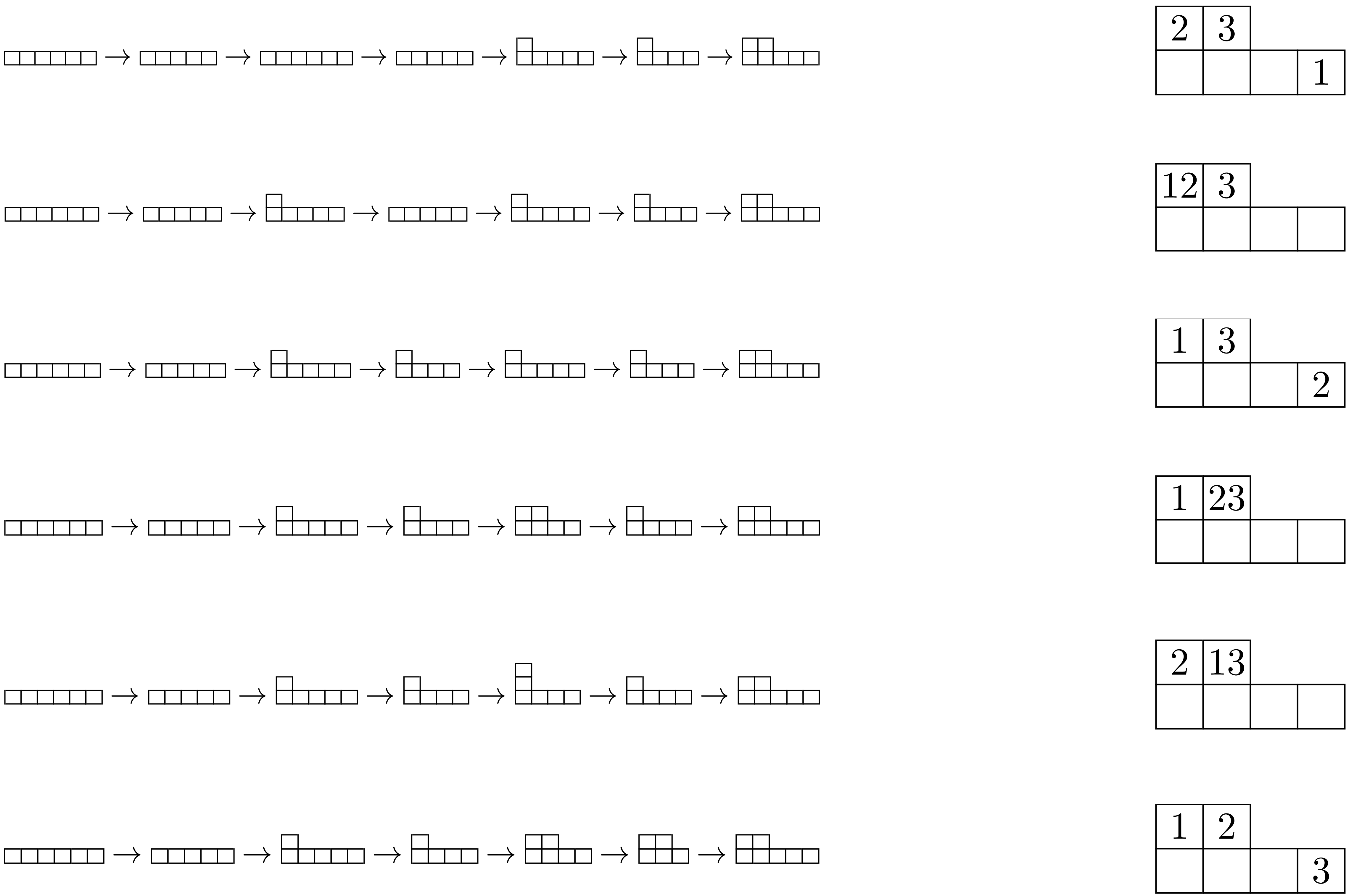
$$f_6 = 1$$
$$\tilde{s}_{()} = h_{()}$$

$$f_6 + f_{51} = 6$$
$$\tilde{s}_{()} + \tilde{s}_1 = h_1$$

$$2f_6 + 3f_{51} + f_{42} + f_{411} = 36$$
$$2\tilde{s}_{()} + 3\tilde{s}_1 + \tilde{s}_2 + \tilde{s}_{11} = h_{11}$$

$$5f_6 + 10f_{51} + 6f_{42} + 6f_{411} + f_{33} + 2f_{321} + f_{3111} = 216$$
$$5\tilde{s}_{()} + 10\tilde{s}_1 + 6\tilde{s}_2 + 6\tilde{s}_{11} + \tilde{s}_3 + \tilde{s}_{21} + \tilde{s}_{111} = h_{111}$$





$$n^k = \sum_{\lambda \vdash n} \left(\begin{array}{c}\text{\# of standard} \\ \text{tableaux shape} \\ \lambda\end{array}\right)\left(\begin{array}{l}\text{\# of standard} \\ \text{set tableaux} \\ \text{in } \{1,2,\dots,k\} \text{ of} \\ \text{shape } \lambda\end{array}\right)$$

Summary.....

- The dimensions of the irreducible partition algebra representations are equal to the number of standard set valued tableaux.
- There is a bijection with the (previously known) combinatorial interpretation (oscillating tableaux) and there is an RSK bijection which explains

$$n^k = \sum_{\lambda \vdash n} \begin{pmatrix} \text{\# of standard} \\ \text{tableaux shape} \\ \lambda \end{pmatrix} \begin{pmatrix} \text{\# of standard} \\ \text{set tableaux} \\ \text{in } \{1,2,\ldots,k\} \text{ of} \\ \text{shape } \lambda \end{pmatrix}$$

$$\begin{matrix} \text{\# of set partitions} \\ \text{of } \{1,2,\ldots,2n\} \end{matrix} = \sum_{|\lambda| \leq n} \begin{pmatrix} \text{\# of standard set} \\ \text{valued tableaux in} \\ \{1,2,\ldots,n\} \text{ of shape } (k,\lambda) \end{pmatrix}^2$$